

Embedding Description Logic Programs into Default Logic

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Abstract

Description logic programs (dl-programs) under the answer set semantics formulated by Eiter *et al.* have been considered as a prominent formalism for integrating rules and ontology knowledge bases. A question of interest has been whether dl-programs can be captured in a general formalism of nonmonotonic logic. In this paper, we study the possibility of embedding dl-programs into default logic. We show that dl-programs under the strong and weak answer set semantics can be embedded in default logic by combining two translations, one of which eliminates the constraint operator from nonmonotonic dl-atoms and the other translates a dl-program into a default theory. For dl-programs without nonmonotonic dl-atoms but with the negation-as-failure operator, our embedding is polynomial, faithful, and modular. In addition, our default logic encoding can be extended in a simple way to capture recently proposed weakly well-supported answer set semantics, for arbitrary dl-programs. These results reinforce the argument that default logic can serve as a fruitful foundation for query-based approaches to integrating ontology and rules. With its simple syntax and intuitive semantics, plus available computational results, default logic can be considered an attractive approach to integration of ontology and rules.

1. Introduction

Logic programming under the answer set semantics (ASP) has been recognized as an expressive nonmonotonic reasoning framework for declarative problem solving and knowledge representation (Marek & Truszczyński, 1999; Niemelä, 1999). Recently, there has been an extensive interest in combining ASP with other logics or reasoning mechanisms. One of the main interests in this direction is the integration of ASP with description logics (DLs) for the Semantic Web. This is due to the fact that, although ontologies expressed in DLs and rules in ASP are two prominent knowledge representation formalisms, each of them has limitations on its own. As (most) DLs are fragments of (many sorted) first order logic, they do not support default, typicality, or nonmonotonic reasoning in general. On the other hand, though there are some recent attempts to extend ASP beyond propositional logic, the core, effective reasoning methods are designed essentially for computation of ground programs; in particular, ASP typically does not reason with unbounded or infinite domains, nor does it support quantifiers. An integration of the two can offer features of both.

A number of proposals for integrating ontology and (nonmonotonic) rules have been put forward (de Bruijn, Eiter, Polleres, & Tompits, 2007; de Bruijn, Pearce, Polleres, & Valverde, 2007;

Eiter, Ianni, Lukasiewicz, Schindlauer, & Tompits, 2008; Motik & Rosati, 2010; Rosati, 2005, 2006; Analyti, Antoniou, & Damásio, 2011; Lukasiewicz, 2010; Lee & Palla, 2011). The existing approaches can be roughly classified into three categories. In the first, typically a nonmonotonic formalism is adopted which naturally embodies both first-order logic and rules, where ontology and rules are written in the same language resulting in a tight coupling (de Bruijn, Eiter, Polleres, & Tompits, 2011; Motik & Rosati, 2010; Lukasiewicz, 2010). The second is a loose approach: an ontology knowledge base and rules share the same constants but not the same predicates, and inference-based communication is via a well-designed interface, called *dl-atoms* (Eiter et al., 2008). In the third approach, rules are treated as hybrid formulas where in model building the predicates in the language of the ontology are interpreted classically, whereas those in the language of rules are interpreted nonmonotonically (Rosati, 2005, 2006; de Bruijn et al., 2007).

The loose coupling approach above stands out as quite unique and it possesses some advantages. In many practical situations, we would like to combine existing knowledge bases, possibly under different logics. In this case, a notion of interface is natural and necessary. The formulation of *dl-programs* adopts such interfaces to ontology knowledge bases. It is worth noticing that *dl-programs* share many similarities with another recent interesting formalism, called *nonmonotonic multi-context systems*, in which knowledge bases under arbitrary logics communicate through *bridge rules* (Brewka & Eiter, 2007).

Informally, a *dl-program* is a pair (O, P) , where O is an ontology knowledge base expressed in a description logic, and P a logic program, where rule bodies may contain queries to the knowledge base O , called *dl-atoms*. Such queries allow to specify inputs from a logic program to the ontology knowledge base. In more detail, a *dl-atom* is of the form

$$DL[S_1 \text{ op}_1 p_1, \dots, S_m \text{ op}_m p_m; Q](\vec{t})$$

where $Q(\vec{t})$ is a query to O , and for each i ($1 \leq i \leq m$), S_i is a concept or a role in O , p_i is a predicate symbol in P having the same arity as S_i , and the operator $\text{op}_i \in \{\oplus, \odot, \ominus\}$. Intuitively, \oplus (resp., \odot) increases S_i (resp., $\neg S_i$) by the extension of p_i , while \ominus (called the *constraint operator*) constrains S_i to p_i , i.e., for an expression $S \ominus p$, for any tuple of constants \vec{t} , in the absence of $p(\vec{t})$ we infer $\neg S(\vec{t})$. Eiter *et al.* proposed weak and strong answer sets for *dl-programs* (Eiter et al., 2008), which were further investigated from the perspective of loop formulas (Wang, You, Yuan, & Shen, 2010) and from the perspective of the logic of here-and-there (Fink & Pearce, 2010).

The interest in *dl-programs* is also due to a technical aspect - it has been a challenging task to embed *dl-programs* into a general nonmonotonic logic. For example, MKNF (Lifschitz, 1991) is arguably among the most expressive and versatile formalisms for integrating rules and description logic knowledge bases (Motik & Rosati, 2010). Although Motik and Rosati were able to show a polynomial embedding of a number of other integration formalisms into MKNF, for *dl-programs* they only showed that if a *dl-program* does not contain the constraint operator \ominus , then it can be translated to a (hybrid) MKNF knowledge base while preserving its strong answer sets.¹ The embedding into quantified equilibrium logic in (Fink & Pearce, 2010) is under the assumption that all *dl-atoms* containing an occurrence of \ominus are nonmonotonic. They do not deal with the case when a *dl-atom* involving \ominus may be monotonic. The embedding into first-order autoepistemic logic (AEL) is under the weak answer set semantics (de Bruijn, Eiter, & Tompits, 2008). For the strong answer

1. The theorem given in (Motik & Rosati, 2010) (Theorem 7.6) only claims to preserve satisfiability. In a personal communication with Motik, it is confirmed that the proof of the theorem indeed establishes a one-to-one correspondence.

set semantics, it is obtained by an embedding of MKNF into first-order autoepistemic logic together with the embedding of dl-programs into MKNF. Thus it only handles the dl-programs without the constraint operator.

In this paper, we investigate the possibility of embedding dl-programs into default logic (Reiter, 1980), under various notions of answer set semantics. Our interest in default logic is due to the fact that it is one of the dominant nonmonotonic formalisms, yet despite the fact that default logic naturally accommodates first-order logic and rules (defaults), curiously it has not been considered explicitly as a framework for integrating ontology and rules. Since the loose approach can be viewed as query-based, the question arises as whether default logic can be viewed as a foundation for query-based approaches to integration of ontologies and rules.

We shall note that the problem of embedding dl-programs into default logic is nontrivial. In fact, given the difficulties in dealing with dl-programs by other expressive nonmonotonic logics, one can expect great technical subtlety in this endeavor. Especially, the treatment of equality is a nontrivial issue.

A main technical result of this paper is that dl-programs can be translated to default theories while preserving their strong and weak answer sets. This is achieved in two steps. In the first, we investigate the operators in dl-programs and observe that the constraint operator \ominus is the only one causing a dl-atom to be nonmonotonic, and a dl-atom may still be monotonic even though it mentions the constraint operator \ominus . To eliminate \ominus from nonmonotonic dl-atoms, we propose a translation π and show that, given a dl-program \mathcal{K} , the strong and weak answer sets of \mathcal{K} correspond exactly to the strong and weak answer sets of $\pi(\mathcal{K})$, respectively, i.e., when restricted to the language of \mathcal{K} , the strong and weak answer sets of $\pi(\mathcal{K})$ are precisely those of \mathcal{K} , and vice versa. An immediate consequence of this result is that it improves a result of (Motik & Rosati, 2010), in that we now know that a much larger class of dl-programs, the class of *normal dl-programs*, can be translated to MKNF knowledge bases, where a dl-program is normal if it has no monotonic dl-atoms that mention the constraint operator \ominus .

For the weak answer set semantics, the translation above can be relaxed so that all dl-atoms containing \ominus can be translated uniformly, and the resulting translation is polynomial. However, for the strong answer set semantics, the above translation relies on the knowledge whether a dl-atom is monotonic or not. In this paper, we present a number of results regarding the upper and lower bounds of determining this condition for description logics *SHIF* and *SHOIN* (Eiter et al., 2008). These results have a broader implication as they apply to the work of (Fink & Pearce, 2010) in embedding dl-programs under strong answer sets into quantified equilibrium logic.

In the second step, we present two approaches to translating dl-programs to default theories in a polynomial, faithful, and modular manner (Janhunen, 1999).² The difference between the two is on the handling of inconsistent ontology knowledge bases. In the first one, an inconsistent ontology knowledge base trivializes the resulting default theory, while following the spirit of dl-programs, in the second approach nontrivial answer sets may still exist in the case of an inconsistent ontology knowledge base. We show that, for a dl-program \mathcal{K} without nonmonotonic dl-atoms, there is a one-to-one correspondence between the strong answer sets of \mathcal{K} and the extensions of its corresponding default theory (whenever the underlying knowledge base is consistent for the first approach). This, along with the result given in the first step, shows that dl-programs under the strong answer set semantics can be embedded into default logic.

2. This means a polynomial time transformation that preserves the intended semantics, uses the symbols of the original language, and translates parts (modules) of the given dl-program independently of each other.

It has been argued that some strong answers may incur self-supports. To overcome this blemish, weakly and strongly well-supported answer set semantics are recently proposed (Shen, 2011). Surprisingly, dl-programs under the weakly well-supported semantics can be embedded into default logic by a small enhancement to our approach in the second step above, and the resulting translation is again polynomial, faithful and modular. Furthermore, if nonmonotonic dl-atoms do not appear in the scope of the default negation *not*, the strongly well-supported semantics coincides with the weakly well-supported semantics. Since default negation already provides a language construct to express default inferences, it can be argued that one need not use the constraint operator \ominus inside it. In this sense, our default logic encoding captures the strongly well-supported semantics as well.

We note that, in embedding dl-programs without nonmonotonic dl-atoms into default logic, one still can use the negation-as-failure operator *not* in dl-programs to express nonmonotonic inferences. The same assumption was adopted in defining a well-founded semantics for dl-programs (Eiter, Lukasiewicz, Ianni, & Schindlauer, 2011). Under this assumption, all the major semantics for dl-programs coincide, and they all can be embedded into default logic by a polynomial, faithful, and modular translation. Thus, the results of this paper not only reveal insights and technical subtleties in capturing dl-programs under various semantics by default logic, but also strengthen the prospect that the latter can serve as a foundation for query-based integration of rules and ontologies.

The main advantage of using default logic to characterize integration of ontology and rules in general, and semantics of dl-programs in particular, is its simple syntax and intuitive semantics, which has led to a collection of computational results in the literature (see, e.g., (Li & You, 1992; Cholewiński, Marek, Mikitiuk, & Truszczyński, 1999; Nicolas, Saubion, & Stéphan, 2001; Chen, Wan, Zhang, & Zhou, 2010)). Interestingly, the more recent effort is on applying ASP techniques to compute default extensions. As long as defaults can be finitely grounded, which is the case for the approach of this paper, these techniques can be extended by combining an ASP-based default logic engine with a description logic reasoner, with the latter being applied as a black box. In contrast, the computational issues are completely absent in the approach under AEL (de Bruijn et al., 2008), and only addressed briefly at an abstract level for the approach based on MKNF (Motik & Rosati, 2010). Furthermore, the representation of dl-programs in default logic leads to new insights in computation for dl-programs, one of which is that the iterative construction of default extensions provides a direct support to well-supportedness for answer sets, so that justifications for positive dependencies can be realized for free.

The main contributions of this paper are summarized as follows.

- We show that dl-programs under the weak and strong answer set semantics can be faithfully and modularly rewritten without constraint operators. The rewriting is polynomial for the weak answer set semantics.
- To embed arbitrary dl-programs into default logic, we present faithful and modular (Janhunen, 1999) translations for the strong answer set semantics, the weak answer set semantics and the weakly well-supported semantics. The translations are also polynomial for the latter two semantics.
- For the strong answer set semantics, the embedding depends on the knowledge of monotonicity of dl-atoms and is polynomial relative to this knowledge, i.e., if the set of monotonic dl-atoms is known. In general, determining this set is intractable; as we show, determining whether a dl-atom is monotonic is EXP-complete under the description logic *SHIF* and

p^{NEXP} -complete under the description logic *SHOIN* (and thus not more expensive than deciding the existence of some strong or weak answer set of a dl-program under these description logics).

- For the two semantics for which we do not provide a polynomial embedding, namely the strong answer set semantics and the strongly well-supported semantics, there are broad classes of dl-programs for which a polynomial embedding can be easily inferred from our results. For the class of dl-programs where nonmonotonic dl-atoms do not appear in the scope of default negation *not*, our embedding is polynomial, faithful, and modular under the strongly well-supported semantics; and for the class of dl-programs where the constraint operator does not appear in a positive dl-atom in rules, our embedding is again polynomial, faithful, and modular under the strong answer set semantics.

The paper is organized as follows. In the next section, we recall the basic definitions of description logics and dl-programs. In Section 3, we present a transformation to eliminate the constraint operator from nonmonotonic dl-atoms. In Section 4, we give transformations from dl-programs to default theories, followed by Sections 5 and 6 on related work and concluding remarks respectively.

2. Preliminaries

In this section, we briefly review the basic notations for description logics (Baader, Calvanese, McGuinness, Nardi, & Patel-Schneider, 2007) and description logic programs (Eiter et al., 2008).

2.1 Description logics

Description Logics are a family of class-based (concept-based) knowledge representation formalisms. We assume a set \mathbf{E} of *elementary datatypes* and a set \mathbf{V} of *data values*. A *datatype theory* $\mathbf{D} = (\Delta^{\mathbf{D}}, \cdot^{\mathbf{D}})$ consists of a *datatype* (or *concrete*) *domain* $\Delta^{\mathbf{D}}$ and a mapping $\cdot^{\mathbf{D}}$ that assigns to every elementary datatype a subset of $\Delta^{\mathbf{D}}$ and to every data value an element of $\Delta^{\mathbf{D}}$. Let $\Psi = (\mathbf{A} \cup \mathbf{R}_A \cup \mathbf{R}_D, \mathbf{I} \cup \mathbf{V})$ be a vocabulary, where \mathbf{A} , \mathbf{R}_A , \mathbf{R}_D , and \mathbf{I} are pairwise disjoint (denumerable) sets of *atomic concepts*, *abstract roles*, *datatype* (or *concrete*) *roles*, and *individuals*, respectively.

A *role* is an element of $\mathbf{R}_A \cup \mathbf{R}_A^- \cup \mathbf{R}_D$, where \mathbf{R}_A^- means the set of inverses of all $R \in \mathbf{R}_A$. *Concepts* are inductively defined as: (1) every atomic concept $C \in \mathbf{A}$ is a concept, (2) if o_1, o_2, \dots are individuals from \mathbf{I} , then $\{o_1, o_2, \dots\}$ is a concept (called *oneOf*), (3) if C and D are concepts, then also $(C \sqcap D)$, $(C \sqcup D)$, and $\neg C$ are concepts (called *conjunction*, *disjunction*, and *negation* respectively). (4) if C is a concept, R is an abstract role from $\mathbf{R}_A \cup \mathbf{R}_A^-$, and n is a nonnegative integer, then $\exists R.C$, $\forall R.C$, $\geq nR$, and $\leq nR$ are concepts (called *exists*, *value*, *atleast*, and *atmost restriction*, respectively), (5) if D is a datatype, U is a datatype role from \mathbf{R}_D , and n is a nonnegative integer, then $\exists U.D$, $\forall U.D$, $\geq nU$, and $\leq nU$ are concepts (called *datatype exists*, *value*, *atleast*, and *atmost restriction*, respectively).

An *axiom* is an expression of one of the forms: (1) $C \sqsubseteq D$, called *concept inclusion axiom*, where C and D are concepts; (2) $R \sqsubseteq S$, called *role inclusion axiom*, where either $R, S \in \mathbf{R}_A$ or $R, S \in \mathbf{R}_D$; (3) $\text{Trans}(R)$, called *transitivity axiom*, where $R \in \mathbf{R}_A$; (4) $C(a)$, called *concept membership axiom*, where C is a concept and $a \in \mathbf{I}$; (5) $R(a, b)$ (resp., $U(a, v)$), called *role membership axiom* where $R \in \mathbf{R}_A$ (resp., $U \in \mathbf{R}_D$) $a, b \in \mathbf{I}$ (resp., $a \in \mathbf{I}$ and v is a data value), (6) $a \approx b$ (resp., $a \not\approx b$), called *equality* (resp., *inequality*) *axiom* where $a, b \in \mathbf{I}$.

A *description logic (DL) knowledge base* O is a finite set of axioms. The $\mathcal{SHOIN}(\mathbf{D})$ *knowledge base* consists of a finite set of above axioms, while the $\mathcal{SHIF}(\mathbf{D})$ *knowledge base* is the one of $\mathcal{SHOIN}(\mathbf{D})$, but without the *oneOf* constructor and with the *atleast* and *atmost* constructors limited to 0 and 1.

The semantics of the two description logics are defined in terms of general first-order interpretations. An *interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ with respect to a datatype theory $\mathbf{D} = (\Delta^{\mathbf{D}}, \cdot^{\mathbf{D}})$ consists of a nonempty (abstract) *domain* $\Delta^{\mathcal{I}}$ disjoint from $\Delta^{\mathbf{D}}$, and a mapping $\cdot^{\mathcal{I}}$ that assigns to each atomic concept $C \in \mathbf{A}$ a subset of $\Delta^{\mathcal{I}}$, to each individual $o \in \mathbf{I}$ an element of $\Delta^{\mathcal{I}}$, to each abstract role $R \in \mathbf{R}_A$ a subset of $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, and to each datatype role $U \in \mathbf{R}_D$ a subset of $\Delta^{\mathcal{I}} \times \Delta^{\mathbf{D}}$. The mapping $\cdot^{\mathcal{I}}$ is extended to all concepts and roles as usual (where $\#S$ denotes the cardinality of a set S):

- $(R^-)^{\mathcal{I}} = \{(a, b) | (b, a) \in R^{\mathcal{I}}\};$
- $\{o_1, \dots, o_n\}^{\mathcal{I}} = \{o_1^{\mathcal{I}}, \dots, o_n^{\mathcal{I}}\};$
- $(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}, (C \sqcup D)^{\mathcal{I}} = C^{\mathcal{I}} \cup D^{\mathcal{I}}, (-C)^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}};$
- $(\exists R.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} | \exists y : (x, y) \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\};$
- $(\forall R.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} | \forall y : (x, y) \in R^{\mathcal{I}} \rightarrow y \in C^{\mathcal{I}}\};$
- $(\geq nR)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} | \#(\{y | (x, y) \in R^{\mathcal{I}}\}) \geq n\};$
- $(\leq nR)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} | \#(\{y | (x, y) \in R^{\mathcal{I}}\}) \leq n\};$
- $(\exists U.D)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} | \exists y : (x, y) \in U^{\mathcal{I}} \wedge y \in D^{\mathbf{D}}\};$
- $(\forall U.D)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} | \forall y : (x, y) \in U^{\mathcal{I}} \rightarrow y \in D^{\mathbf{D}}\};$
- $(\geq nU)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} | \#(\{y | (x, y) \in U^{\mathcal{I}}\}) \geq n\};$
- $(\leq nU)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} | \#(\{y | (x, y) \in U^{\mathcal{I}}\}) \leq n\}.$

Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation respect to $\mathbf{D} = (\Delta^{\mathbf{D}}, \cdot^{\mathbf{D}})$, and F an axiom. We say that \mathcal{I} *satisfies* F , written $\mathcal{I} \models F$, is defined as follows: (1) $\mathcal{I} \models C \sqsubseteq D$ iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$; (2) $\mathcal{I} \models R \sqsubseteq S$ iff $R^{\mathcal{I}} \subseteq S^{\mathcal{I}}$; (3) $\mathcal{I} \models \text{Trans}(R)$ iff $R^{\mathcal{I}}$ is transitive; (4) $\mathcal{I} \models C(a)$ iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$; (5) $\mathcal{I} \models R(a, b)$ iff $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$ (resp., $\mathcal{I} \models U(a, v)$ iff $(a^{\mathcal{I}}, v^{\mathbf{D}}) \in U^{\mathcal{I}}$); (6) $\mathcal{I} \models a \approx b$ iff $a^{\mathcal{I}} = b^{\mathcal{I}}$ (resp., $\mathcal{I} \models a \not\approx b$ iff $a^{\mathcal{I}} \neq b^{\mathcal{I}}$). \mathcal{I} *satisfies* a DL knowledge base O , written $\mathcal{I} \models O$, if $\mathcal{I} \models F$ for any $F \in O$. In this case, we call \mathcal{I} a *model* of O . An axiom F is a *logical consequence* of a DL knowledge base O , written $O \models F$, if any model of O is also a model of F .

2.2 Description logic programs

Let $\Phi = (\mathcal{P}, \mathcal{C})$ be a first-order vocabulary with nonempty finite sets \mathcal{C} and \mathcal{P} of constant symbols and predicate symbols respectively such that \mathcal{P} is disjoint from $\mathbf{A} \cup \mathbf{R}$ and $\mathcal{C} \subseteq \mathbf{I}$. *Atoms* are formed from the symbols in \mathcal{P} and \mathcal{C} as usual.

A *dl-atom* is an expression of the form

$$DL[S_1 \text{ op}_1 p_1, \dots, S_m \text{ op}_m p_m; Q](\vec{t}), \quad (m \geq 0) \quad (1)$$

where

- each S_i is either a concept, a role or its negation,³ or a special symbol in $\{\approx, \not\approx\}$;
- $op_i \in \{\oplus, \odot, \ominus\}$ (we call \ominus the *constraint operator*);
- p_i is a unary predicate symbol in \mathcal{P} if S_i is a concept, and a binary predicate symbol in \mathcal{P} otherwise. The p_i 's are called *input predicate symbols*;
- $Q(\vec{t})$ is a *dl-query*, i.e., either (1) $C(t)$ where $\vec{t} = t$; (2) $C \sqsubseteq D$ where \vec{t} is an empty argument list; (3) $R(t_1, t_2)$ where $\vec{t} = (t_1, t_2)$; (4) $t_1 \approx t_2$ where $\vec{t} = (t_1, t_2)$; or their negations, where C and D are concepts, R is a role, and \vec{t} is a tuple of constants.

The precise meanings of $\{\oplus, \odot, \ominus\}$ will be defined shortly. Intuitively, $S \oplus p$ extends S by the extension of p . Similarly, $S \odot p$ extends $\neg S$ by the extension of p , and $S \ominus p$ constrains S to p . A *dl-rule* (or simply a *rule*) is an expression of the form

$$A \leftarrow B_1, \dots, B_m, \text{not } B_{m+1}, \dots, \text{not } B_n, (n \geq m \geq 0) \quad (2)$$

where A is an atom, each B_i ($1 \leq i \leq n$) is an atom⁴ or a dl-atom. We refer to A as its *head*, while the conjunction of B_i ($1 \leq i \leq m$) and $\text{not } B_j$ ($m+1 \leq j \leq n$) is its *body*. For convenience, we abbreviate a rule in the form (2) as

$$A \leftarrow \text{Pos}, \text{not Neg} \quad (3)$$

where $\text{Pos} = \{B_1, \dots, B_m\}$ and $\text{Neg} = \{B_{m+1}, \dots, B_n\}$. Let r be a rule of the form (3). If $\text{Neg} = \emptyset$ and $\text{Pos} = \emptyset$, r is a *fact* and we may write it as “ A ” instead of “ $A \leftarrow$ ”. A *description logic program (dl-program)* $\mathcal{K} = (O, P)$ consists of a DL knowledge base O and a finite set P of dl-rules. In what follows we assume the vocabulary of P is implicitly given by the constant symbols and predicate symbols occurring in P , \mathcal{C} consists of the constants occurring in atoms of P , and P is grounded (no atoms containing variables) unless stated otherwise.

Given a dl-program $\mathcal{K} = (O, P)$, the *Herbrand base* of P , denoted by HB_P , is the set of atoms occurring in P and the ones formed from the predicate symbols of \mathcal{P} occurring in some dl-atoms of P and the constant symbols in \mathcal{C} .⁵ It is clear that HB_P is in polynomial size of \mathcal{K} . An *interpretation* I (relative to P) is a subset of HB_P . Such an I is a *model* of an atom or dl-atom A under O , written $I \models_O A$, if the following holds:

- if $A \in HB_P$, then $I \models_O A$ iff $A \in I$;
- if A is a dl-atom $DL(\lambda; Q)(\vec{t})$ of the form (1), then $I \models_O A$ iff $O(I; \lambda) \models Q(\vec{t})$ where $O(I; \lambda) = O \cup \bigcup_{i=1}^m A_i(I)$ and, for $1 \leq i \leq m$,

$$A_i(I) = \begin{cases} \{S_i(\vec{e}) \mid p_i(\vec{e}) \in I\}, & \text{if } op_i = \oplus; \\ \{\neg S_i(\vec{e}) \mid p_i(\vec{e}) \in I\}, & \text{if } op_i = \odot; \\ \{\neg S_i(\vec{e}) \mid p_i(\vec{e}) \notin I\}, & \text{if } op_i = \ominus; \end{cases}$$

3. We allow negation of a role for convenience, so that we can replace “ $S \odot p$ ” with an equivalent form “ $\neg S \oplus p$ ” in dl-atoms. The negation of a role is not explicitly present in (Eiter et al., 2008). As discussed there, negative role assertions can be emulated in *SHLF* and *SHOLN* (and in fact also in *ALC*).

4. Different from that of (Eiter et al., 2008), we consider ground atoms instead of literals for convenience.

5. Note that this slightly deviates from the usual convention of the Herbrand base; ground atoms that are not in the Herbrand base as considered here are always false in answer sets.

where \vec{e} is a tuple of constants over \mathcal{C} . As we allow negation of role, $S \odot p$ can be replaced with $\neg S \oplus p$ in any dl-atom. In addition, we can shorten $S_1 \text{ op } p, \dots, S_k \text{ op } p$ as $(S_1 \sqcup \dots \sqcup S_k) \text{ op } p$ where $S_i \text{ op } p$ appears in λ for all i ($1 \leq i \leq k$) and $\text{op} \in \{\oplus, \odot, \ominus\}$. Thus dl-atoms can be equivalently rewritten into ones without using the operator \odot , and every predicate p appears at most once for each operator \oplus and \ominus . For instance, the dl-atom $DL[S_1 \oplus p, S_2 \oplus p, S_1 \ominus p, S_2 \ominus p, Q](\vec{t})$ can be equivalently written as $DL[(S_1 \sqcup S_2) \oplus p, (S_1 \sqcup S_2) \ominus p, Q](\vec{t})$.

An interpretation $I \subseteq HB_P$ is a *model* of “not A ”, written $I \models_O \text{not } A$, if I is not a model of A , i.e., $I \not\models_O A$. The interpretation I is a *model* of a dl-rule of the form (3) iff $I \models_O B$ for any $B \in Pos$ and $I \not\models_O B'$ for any $B' \in Neg$ implies that $I \models_O A$. An interpretation I is a *model* of a dl-program $\mathcal{K} = (O, P)$, written $I \models_O \mathcal{K}$, iff I is a model of each rule of P .

2.2.1 MONOTONIC DL-ATOMS

A dl-atom A is *monotonic* (relative to a dl-program $\mathcal{K} = (O, P)$) if $I \models_O A$ implies $I' \models_O A$, for all I' such that $I \subseteq I' \subseteq HB_P$, otherwise A is *nonmonotonic*. It is clear that if a dl-atom does not mention the constraint operator then it is monotonic. However, a dl-atom may be monotonic even if it mentions the constraint operator. For example, the dl-atom $DL[S \odot p, S \ominus p; \neg S](a)$ is a tautology (which is monotonic).

Evidently, the constraint operator is the only one that may cause a dl-atom to be nonmonotonic. This sufficient condition for monotonicity can be efficiently checked; for the case where the constraint operator may appear, the following generic upper bound on complexity is easily derived. We refer to the *query complexity* of a ground dl-atom A of form (1) in \mathcal{K} as the complexity of deciding, given $K = (O, P)$, A , and an arbitrary interpretation I , whether $O(I; \lambda) \models A$ holds.

Proposition 1 *Let $\mathcal{K} = (O, P)$ be a (ground) dl-program, and A be a dl-atom occurring in P which has query complexity in class C . Then deciding whether A is monotonic is in co-NP^C .*

Proof: Indeed, to show that A of form (1) is nonmonotonic, one can guess restrictions I_A and I'_A of interpretations I and I' , respectively, to the predicates occurring in A such that $I_A \subseteq I'_A$ and $I_A \models_O A$ but $I'_A \not\models_O A$ (clearly, $J \models_O A$ iff $J_A \models_O A$ for arbitrary interpretations J). The guess for I_A and I'_A is of polynomial size in the size of \mathcal{K} (assuming that the set of constants \mathcal{C} is explicit in \mathcal{K} , or is constructible in polynomial time), and preparing $O(I_A; \lambda)$ and $O(I'_A; \lambda)$ is feasible in polynomial time (in fact, easily in logarithmic space). Using the oracle, we can decide $O(I_A; \lambda) \models Q(\vec{t})$ and $O(I'_A; \lambda) \models Q(\vec{t})$, and thus $I_A \models_O A$ but $I'_A \not\models_O A$. Overall, the complexity is in co-NP^C . ■

Depending on the underlying description logic, this upper bound might be lower or complemented by a matching hardness result. In fact, for *SHIF* and *SHOIN*, the latter turns out to be the case. DL-atoms over these description logics have a query complexity that is complete for $C = \text{EXP}$ and $C = \text{NEXP}$, respectively. By employing well-known identities of complexity classes, we obtain the following result.

Theorem 1 *Given a (ground) dl-program $\mathcal{K} = (O, P)$ and a dl-atom A occurring in P , deciding whether A is monotonic is (i) EXP -complete, if O is a *SHIF* knowledge base and (ii) P^{NEXP} -complete, if O is a *SHOIN* knowledge base.*

Proof: The membership part for (i) $SHIF$ follows easily from Proposition 1 and the fact that $SHIF$ has query complexity in EXP; indeed, each dl-query evaluation $O(I; \lambda) \models_O$ can be transformed in polynomial time to deciding satisfiability of a $SHIF$ knowledge base, which is EXP-complete in general (Tobies, 2001; Horrocks & Patel-Schneider, 2003). Now $\text{co-NP}^{\text{EXP}} = \text{EXP} = \text{NP}^{\text{EXP}}$; indeed, the computation tree of a nondeterministic Turing machine with polynomial running time and EXP oracle access has single exponential (in the input size) many nodes, which can be traversed in exponential time; simulating an oracle call in a node is possible in exponential time in the size of the (original) input. Overall, this yields an exponential time upper bound.

The membership part for (ii) $SHOIN$ follows analogously from Proposition 1 and the fact that $SHOIN$ has co-NEXP-complete query complexity, which follows from NEXP-completeness of the knowledge base satisfiability problem in $SHOIN$ (for both unary and binary number encoding; see (Horrocks & Patel-Schneider, 2003; Pratt-Hartmann, 2005)). Now $\text{co-NP}^{\text{co-NEXP}} = \text{co-NP}^{\text{NEXP}} = \text{P}^{\text{NEXP}} (= \text{NP}^{\text{NEXP}})$; here the second equality holds by results in (Hemachandra, 1989).

The hardness parts for (i) and (ii) are shown by reductions of suitable EXP- resp. P^{NEXP} -complete problems, building on constructions in (Eiter et al., 2008) (see Appendix A). ■

For convenience, we use DL_P to denote the set of all dl-atoms that occur in P , $DL_P^+ \subseteq DL_P$ to denote the set of monotonic dl-atoms, and $DL_P^? = DL_P \setminus DL_P^+$. Note that this is different from that of (Eiter et al., 2008) where DL_P^+ is assumed to be a set of ground dl-atoms in DL_P which are known to be monotonic, while $DL_P^?$ denotes the set of remaining dl-atoms. Thus $DL_P^?$ is allowed to contain monotonic dl-atoms as well in (Eiter et al., 2008). Our definition represents the ideal situation where monotonicity can be finitely verified, which is the case for decidable description logic knowledge bases. Note also that by Theorem 1, for $SHIF$ and $SHOIN$ knowledge bases computing DL_P^+ is possible with no respectively mild complexity increase compared to basic reasoning tasks in the underlying description logic.

2.2.2 SOME CLASSES OF DL-PROGRAMS

A dl-program $\mathcal{K} = (O, P)$ is *positive*, if (i) P is “not”-free, and (ii) every dl-atom is monotonic relative to \mathcal{K} . Positive dl-programs have attractive semantics properties; e.g., it is evident that a positive dl-program \mathcal{K} has a (set inclusion) least model.

From the results above, we easily obtain the following results on recognizing positive dl-programs.

Proposition 2 *Deciding whether a given (not necessarily ground) dl-program $\mathcal{K} = (O, P)$ is positive is in co-NP^C , if every dl-atom in the ground version of P has query complexity in C .*

Proof: \mathcal{K} is not positive if either (i) P is not “not”-free, which can be checked in polynomial time, or (ii) some dl-atom A in the ground version of P is nonmonotonic; such an A can be guessed and verified, by the hypothesis, in polynomial time with an oracle for C ; hence the result. ■

Theorem 2 *Deciding whether a given (not necessarily ground) dl-program $\mathcal{K} = (O, P)$ is positive is (i) EXP-complete, if O is a $SHIF$ knowledge base and (ii) P^{NEXP} -complete, if O is a $SHOIN$ knowledge base.*

Proof: The membership parts are immediate from Proposition 2, and the hardness parts from the hardness proofs in Theorem 1: the atom A is monotonic relative to the constructed dl-program \mathcal{K}

iff \mathcal{K} is positive. ■

Thus, the test whether a dl-program is positive (and similarly, whether all dl-atoms in it are monotonic) for *SHIF* and *SHOLN* knowledge bases is also not expensive compared to basic reasoning tasks.

Besides positive dl-programs, another important subclass are *canonical dl-programs*, where a dl-program $\mathcal{K} = (O, P)$ is *canonical*, if P mentions no constraint operator. Clearly, canonical dl-programs are easy to recognize. The same holds for the more general class of *normal dl-programs*, where a dl-program $\mathcal{K} = (O, P)$ is *normal*, if no monotonic dl-atom occurs in P that mentions the constraint operator. Note that normal dl-programs are not positive in general; since monotonic dl-atoms mentioning the constraint operator are rather exceptional, the normal dl-programs include most dl-programs relevant for practical applications.

Example 1 Consider the following dl-programs, which we will refer to repeatedly in the sequel.

- $\mathcal{K}_1 = (O_1, P_1)$ where $O_1 = \{S \sqsubseteq S'\}$ and $P_1 = \{p(a) \leftarrow DL[S \oplus p; S'](a)\}$. The single dl-atom in P_1 has no constraint operator, and thus \mathcal{K}_1 is canonical (hence also normal); moreover, since ‘not’ does not occur in P_1 , \mathcal{K}_1 is also positive.
- $\mathcal{K}_2 = (O_2, P_2)$ where $O_2 = \emptyset$ and $P_2 = \{p(a) \leftarrow DL[S \oplus p, S' \ominus q; S \sqcap \neg S'](a)\}$. Here, the constraint operator occurs in P_2 , thus \mathcal{K}_2 is not canonical. Furthermore, the single dl-atom in P_2 is nonmonotonic, hence \mathcal{K}_2 is also not positive. However, \mathcal{K}_2 is normal.

2.2.3 STRONG AND WEAK ANSWER SETS

Let $\mathcal{K} = (O, P)$ be a positive dl-program. The immediate consequence operator $\gamma_{\mathcal{K}} : 2^{HB_P} \rightarrow 2^{HB_P}$ is defined as, for any $I \subseteq HB_P$,

$$\gamma_{\mathcal{K}}(I) = \{h \mid h \leftarrow Pos \in P \text{ and } I \models_O A \text{ for any } A \in Pos\}.$$

Since $\gamma_{\mathcal{K}}$ is monotonic, the least fix-point of $\gamma_{\mathcal{K}}$ always exists which is the least model of \mathcal{K} . By $\text{lfp}(\gamma_{\mathcal{K}})$ we denote the least fix-point of $\gamma_{\mathcal{K}}$, which can be iteratively constructed as below:

- $\gamma_{\mathcal{K}}^0 = \emptyset$;
- $\gamma_{\mathcal{K}}^{n+1} = \gamma_{\mathcal{K}}(\gamma_{\mathcal{K}}^n)$.

It is clear that the least fixpoint $\text{lfp}(\gamma_{\mathcal{K}}) = \gamma_{\mathcal{K}}^\infty$.

We are now in the position to recall the semantics of dl-programs. Let $\mathcal{K} = (O, P)$ be a dl-program. The *strong dl-transform* of \mathcal{K} relative to O and an interpretation $I \subseteq HB_P$, denoted by $\mathcal{K}^{s,I}$, is the positive dl-program (O, sP_O^I) , where sP_O^I is obtained from P by deleting:

- the dl-rule r of the form (2) such that either $I \not\models_O B_i$ for some $1 \leq i \leq m$ and $B_i \in DL_P^?$, or $I \models_O B_j$ for some $m+1 \leq j \leq n$; and
- the nonmonotonic dl-atoms and *not* A from the remaining dl-rules where A is an atom or a dl-atom.

The interpretation I is a *strong answer set* of \mathcal{K} if it is the least model of $\mathcal{K}^{s,I}$, i.e., $I = \text{lfp}(\gamma_{\mathcal{K}^{s,I}})$.⁶

The *weak dl-transform* of \mathcal{K} relative to O and an interpretation $I \subseteq \text{HB}_P$, denoted by $\mathcal{K}^{w,I}$, is the positive dl-program (O, wP_O^I) , where wP_O^I is obtained from P by deleting:

- the dl-rules of the form (2) such that either $I \not\models_O B_i$ for some $1 \leq i \leq m$ and $B_i \in \text{DL}_P$, or $I \models_O B_j$ for some $m+1 \leq j \leq n$; and
- the dl-atoms and *not* A from the remaining dl-rules where A is an atom or dl-atom.

The interpretation I is a *weak answer set* of \mathcal{K} if I is the least model of $\mathcal{K}^{w,I}$, i.e., $I = \text{lfp}(\gamma_{\mathcal{K}^{w,I}})$.

The following proposition shows that, given a dl-program $\mathcal{K} = (O, P)$, if O is inconsistent then strong and weak answer sets of \mathcal{K} coincide, and are minimal.

Proposition 3 *Let $\mathcal{K} = (O, P)$ be a dl-program where O is inconsistent and $I \subseteq \text{HB}_P$. Then*

- (i) *I is a strong answer set of \mathcal{K} if and only if I is a weak answer set of \mathcal{K} .*
- (ii) *The strong and weak answer sets of \mathcal{K} are minimal under set inclusion.*

Proof: By the inconsistency of O , it is clear that every dl-atom A occurring in P is monotonic and $M \models_O A$ for any $M \subseteq \text{HB}_P$.

(i) Note that the only difference between sP_O^I and wP_O^I is that there exist some dl-atoms in sP_O^I but not in wP_O^I , i.e., for any dl-rule $r = (h \leftarrow \text{Pos}, \text{not Neg})$ in P , $(h \leftarrow \text{Pos})$ belongs to sP_O^I if and only if $(h \leftarrow \text{Pos}')$ belongs to wP_O^I where $\text{Pos}' = \{h \in \text{HB}_P \mid h \in \text{Pos}\}$. However note that $\emptyset \models_O A$ for any dl-atom $A \in \text{Pos} \setminus \text{Pos}'$. It follows that $\text{lfp}(\gamma_{\mathcal{K}^{s,I}}) = \text{lfp}(\gamma_{\mathcal{K}^{w,I}})$. Consequently I is a strong answer set of \mathcal{K} if and only if I is a weak answer set of \mathcal{K} .

(ii) By Theorem 4.13 of (Eiter et al., 2008), the strong answer sets of \mathcal{K} are minimal. It implies that the weak answer sets of \mathcal{K} are minimal as well by (i) of the proposition. \blacksquare

Example 2 [Continued from Example 1] Reconsider the dl-programs in Example 1.

- The dl-program $\mathcal{K}_1 = (O_1, P_1)$, where $O_1 = \{S \sqsubseteq S'\}$ and $P_1 = \{p(a) \leftarrow \text{DL}[S \oplus p; S'](a)\}$, has a unique strong answer set $I_1 = \emptyset$ and two weak answer sets I_1 and $I_2 = \{p(a)\}$. The interested reader may verify the following: $O_1(I_2; S \oplus p) = O_1 \cup \{S(a)\}$, and clearly $O_1 \not\models S'(a)$ and $\{S(a), S \sqsubseteq S'\} \models S'(a)$. So the weak dl-transformation relative to O_1 and I_2 is $\mathcal{K}_1^{w,I_2} = (O_1, \{p(a) \leftarrow\})$. Since I_2 coincides with the least model of $\{p(a) \leftarrow\}$, it is a weak answer set of \mathcal{K}_1 . Similarly, one can verify that the strong dl-transformation relative to O_1 and I_2 is $\mathcal{K}_1^{s,I_2} = \mathcal{K}_1$. Its least model is the empty set, so I_2 is not a strong answer set of \mathcal{K}_1 .
- For the dl-program $\mathcal{K}_2 = (O_2, P_2)$, where $O_2 = \emptyset$ and $P_2 = \{p(a) \leftarrow \text{DL}[S \oplus p, S' \ominus q; S \sqcap \neg S'](a)\}$, both \emptyset and $\{p(a)\}$ are strong and weak answer sets.

6. Note that, under our notion of $\text{DL}_P^?$, namely $\text{DL}_P^?$ is the set of nonmonotonic dl-atoms w.r.t. a given dl-program, the strong answer set semantics is the strongest among possible variations under the definition of (Eiter et al., 2008), where $\text{DL}_P^?$ may contain monotonic dl-atoms, in that given a dl-program \mathcal{K} , any strong answer set of \mathcal{K} under our definition is a strong answer set of \mathcal{K} under the definition of (Eiter et al., 2008).

These dl-programs show that strong (and weak) answer sets may not be (set inclusion) minimal. It has been shown that if a dl-program contains no nonmonotonic dl-atoms then its strong answer sets are minimal (cf. Theorem 4.13 of (Eiter et al., 2008)). However, this does not hold for weak answer sets as shown by the dl-program \mathcal{K}_1 above, even if it is positive. It has also been shown that strong answer sets are always weak answer sets, but not vice versa. Thus the question rises: is it the case that, for any dl-program \mathcal{K} and interpretation I , if I is a weak answer set of \mathcal{K} , then there is $I' \subseteq I$ such that I' is a strong answer of \mathcal{K} ? We give a negative answer to this question by the following example.

Example 3 Let $\mathcal{K} = (\emptyset, P)$ where P consists of

$$p(a) \leftarrow DL[S \oplus p; S](a), \quad p(a) \leftarrow \text{not } DL[S \oplus p; S](a).$$

Note that \mathcal{K} is canonical and normal, but not positive. Intuitively, P expresses reasoning by cases: regardless of whether the dl-atom $A = DL[S \oplus p; S](a)$ evaluates to false, $p(a)$ should be true. Let $I = \{p(a)\}$. We have that $wP_O^I = \{p(a) \leftarrow\}$, thus I is a weak answer set of \mathcal{K} . However, note that $sP_O^I = \{p(a) \leftarrow DL[S \oplus p; S](a)\}$. The least model of $\mathcal{K}^{s,I}$ is \emptyset ($\neq I$). So that I is not a strong answer set of \mathcal{K} . Now consider $I' = \emptyset$. We have $sP_O^{I'} = \{p(a) \leftarrow DL[S \oplus p; S](a), p(a) \leftarrow\}$. The least model of $\mathcal{K}^{s,I'}$ is $\{p(a)\}$ ($\neq I'$). Thus I' is not a strong answer set of \mathcal{K} . In fact, \mathcal{K} has no strong answer sets at all. This is in line with the intuition that, as $O = \emptyset$ is empty, $p(a)$ can not be foundedly derived without the assumption that $p(a)$ is true.

3. Eliminating the Constraint Operator from Nonmonotonic DL-atoms

Intuitively, translating a nonmonotonic dl-atom into a monotonic is to replace $S \ominus p$ with $S \odot p'$ where p' is a fresh predicate having the same arity as p and p' stands for the negation of p . In what follows, we show that the constraint operator can be eliminated from nonmonotonic dl-atoms while preserving both weak and strong answer sets. As mentioned previously, we assume that the signatures \mathcal{P} and \mathcal{C} are implicitly given for a given dl-program \mathcal{K} . Any predicate symbol not occurring in \mathcal{K} is a fresh one.

Definition 1 ($\pi(\mathcal{K})$) Let $\mathcal{K} = (O, P)$ be a dl-program. We define $\pi(\mathcal{K}) = (O, \pi(P))$ where $\pi(P) = \bigcup_{r \in P} \pi(r)$ and $\pi(r)$, assuming r is of the form (2), consists of

(i) the rule

$$A \leftarrow \pi(B_1), \dots, \pi(B_m), \pi(\text{not } B_{m+1}), \dots, \pi(\text{not } B_n) \quad (4)$$

where

$$\pi(B) = \begin{cases} B, & \text{if } B \text{ is an atom or a monotonic dl-atom;} \\ \text{not } \pi_B, & \text{if } B \text{ is a nonmonotonic dl-atom,} \end{cases}$$

in which π_B is a fresh propositional atom, and

$$\pi(\text{not } B) = \begin{cases} \text{not } B, & \text{if } B \text{ is an atom;} \\ \text{not } DL[\pi(\lambda); Q](\vec{t}), & \text{if } B = DL[\lambda, Q](\vec{t}), \end{cases}$$

where $\pi(\lambda)$ is obtained from λ by replacing each “ $S \ominus p$ ” with “ $S \odot \pi_p$ ”, and π_p is a fresh predicate having the same arity as p ;

(ii) for each nonmonotonic dl-atom $B \in \{B_1, \dots, B_m\}$, the following rule:

$$\pi_B \leftarrow \pi(\text{not } B) \quad (5)$$

where π_B is the same atom as mentioned in (i) and

(iii) for each predicate p such that “ $S \ominus p$ ” occurs in some nonmonotonic dl-atom of r , the instantiations of the rule:

$$\pi_p(\vec{x}) \leftarrow \text{not } p(\vec{x}) \quad (6)$$

where \vec{x} is a tuple of distinct variables matching the arity of p , and π_p is the same predicate as mentioned in (i).

Intuitively, the idea in π is the following. Recall that “ $S \ominus p$ ” means “infer $\neg S(\vec{c})$ in absence of $p(\vec{c})$ ”. Thus if $\pi_p(\vec{c})$ stands for the absence of $p(\vec{c})$ then “ $S \ominus p$ ” should have the same meaning as that of “ $S \odot \pi_p$ ”. Thus, a nonmonotonic dl-atom can be re-expressed by a monotonic dl-atom and “not”. Note that $\pi(P)$ may still contain dl-atoms with the constraint operator, but they are all monotonic dl-atoms.

Example 4 Let us consider the following dl-programs.

- Let $\mathcal{K}_1 = (\emptyset, P_1)$ where P_1 consists of

$$p(a) \leftarrow \text{not } DL[S \ominus p; \neg S](a).$$

Note that \mathcal{K}_1 is normal but neither canonical nor positive. It is not difficult to verify that \mathcal{K}_1 has two weak answer sets \emptyset and $\{p(a)\}$. They are strong answer sets of \mathcal{K}_1 as well. According to the translation π , we have $\pi(\mathcal{K}_1) = (\emptyset, \pi(P_1))$, where $\pi(P_1)$ consists of

$$p(a) \leftarrow \text{not } DL[S \odot \pi_p; \neg S](a), \quad \pi_p(a) \leftarrow \text{not } p(a).$$

It is easy to see that $\pi(\mathcal{K}_1)$ has only two weak answer sets, $\{p(a)\}$ and $\{\pi_p(a)\}$, which are also strong answer sets of $\pi(\mathcal{K}_1)$. They correspond to $\{p(a)\}$ and \emptyset respectively when restricted to HB_{P_1} .

- Let $\mathcal{K}_2 = (\emptyset, P_2)$ where P_2 consists of

$$p(a) \leftarrow \text{not } DL[S \ominus p, S' \odot q, S' \ominus q; \neg S \sqcap \neg S'](a).$$

Recall that the dl-atom $DL[S' \odot q, S' \ominus q; \neg S](a)$ is a tautology, hence monotonic; thus \mathcal{K}_2 is not normal. The strong and weak answer sets of \mathcal{K}_2 are the same as those of \mathcal{K}_1 . Please note that $\pi(P_2)$ consists of

$$\begin{aligned} p(a) &\leftarrow \text{not } DL[S \odot \pi_p, S' \odot q, S' \odot \pi_q; \neg S \sqcap \neg S'](a), \\ \pi_p(a) &\leftarrow \text{not } p(a), \quad \pi_q(a) \leftarrow \text{not } q(a). \end{aligned}$$

The strong and weak answer sets of $\pi(\mathcal{K}_2)$ are $\{\pi_q(a), \pi_p(a)\}$ and $\{\pi_q(a), p(a)\}$. They correspond to \emptyset and $\{p(a)\}$ respectively when restricted to HB_{P_2} .

- Let \mathcal{K}_3 be the dl-program \mathcal{K}_2 in Example 1. Then $\pi(\mathcal{K}_3) = (\emptyset, P')$ where P' consists of

$$\begin{aligned} p(a) &\leftarrow \text{not } \pi_A, & \pi_q(a) &\leftarrow \text{not } q(a), \\ \pi_A &\leftarrow \text{not } DL[S \oplus p, S' \odot \pi_q, S \sqcap \neg S'](a) \end{aligned}$$

where $A = DL[S \oplus p, S' \odot q; S \sqcap \neg S'](a)$. One can check that $\pi(\mathcal{K}_3)$ has two strong answer sets, $\{\pi_q(a), \pi_A\}$ and $\{\pi_q(a), p(a)\}$, which are \emptyset and $\{p(a)\}$ whenever restricted to the original Herbrand base.

The main insight revealed by the translation π is, while a negative dl-atom is rewritten by replacing a \ominus expression by a \odot expression, any positive nonmonotonic dl-atom is negated twice, which emulates “double negation” in nested expressions (Lifschitz, Tang, & Turner, 1999).⁷

Although the translation π provides an interesting characterization, due to the difficulty of checking the monotonicity of a dl-atom, for an arbitrary dl-program the translation can be expensive as it depends on checking the entailment relation over the underlying description logic. However, for the class of normal dl-programs, π takes polynomial time since checking the monotonicity of dl-atoms amounts to checking the existence of the constraint operator, and predicates occurring in dl-atoms have the arity at most 2.

We now proceed to show some properties of the translation π .

For any dl-program \mathcal{K} , $\pi(\mathcal{K})$ has no nonmonotonic dl-atoms left. Thus, by Theorem 4.13 of (Eiter et al., 2008), we have

Proposition 4 *Let \mathcal{K} be a dl-program. If $I \subseteq HB_{\pi(P)}$ is a strong answer set of $\pi(\mathcal{K})$ then I is minimal, i.e., there is no $I' \subset I$ such that I' is a strong answer set of $\pi(\mathcal{K})$.*

Proof: It is evident by Theorem 4.13 of (Eiter et al., 2008) and $DL_{\pi(P)}^? = \emptyset$. ■

The dl-programs in the above example show that the translation π preserves both strong and weak answer sets of a given dl-program in the extended language, i.e., the strong and weak answer sets of $\pi(\mathcal{K})$ are those of \mathcal{K} when restricted to the language of \mathcal{K} . In what follows, we formally build up a one-to-one mapping between answer sets of a dl-program \mathcal{K} and those of $\pi(\mathcal{K})$.

For convenience, given a dl-program $\mathcal{K} = (O, P)$ and $I \subseteq HB_P$, we denote $\pi(I) = I \cup \pi_1(I) \cup \pi_2(I)$ where

$$\begin{aligned} \pi_1(I) &= \{\pi_p(\vec{c}) \in HB_{\pi(P)} \mid p(\vec{c}) \notin I\}, \text{ and} \\ \pi_2(I) &= \{\pi_A \in HB_{\pi(P)} \mid A \in DL_P^? \ \& \ I \not\models_O A\}. \end{aligned}$$

Lemma 1 *Let $\mathcal{K} = (O, P)$ be a dl-program, $I \subseteq HB_P$. Then*

(i) *for any atom A occurring in P*

$$I \models_O A \text{ iff } I \cup \pi_1(I) \models_O \pi(A) \text{ iff } \pi(I) \models_O \pi(A);$$

(ii) *for any dl-atom $A = DL[\lambda; Q](\vec{t})$ occurring in P ,*

$$I \models_O A \text{ iff } I \cup \pi_1(I) \models_O DL[\pi(\lambda); Q](\vec{t}) \text{ iff } \pi(I) \not\models_O \pi(\text{not } A).$$

7. A similar logic treatment has been found in a number of recent approaches to the semantics of various classes of logic programs, e.g., in the “double negation” interpretation of weight constraint programs (Ferraris & Lifschitz, 2005; Liu & You, 2011).

Proof: (i) It is obvious since $\pi(A) = A$ and predicates of the form π_p and π_A do not occur in \mathcal{K} .
 (ii) If there is no constraint operator occurring in λ then $DL[\pi(\lambda); Q](\vec{t}) = DL[\lambda; Q](\vec{t})$. Thus in this case, it is trivial as predicates of the form π_p and π_A do not occur in \mathcal{K} , and $\pi(not A) = not A$.
 Suppose there exists at least one constraint operator in λ . It is clear that $I \cup \pi_1(I) \models_O DL[\pi(\lambda); Q](\vec{t})$ if and only if $\pi(I) \not\models_O \pi(not A)$, and evidently, for any atom $\pi_p(\vec{c}) \in HB_{\pi(P)}$, $\pi_p(\vec{c}) \in \pi_1(I)$ if and only if $p(\vec{c}) \notin I$. For clarity and without loss of generality, let $\lambda = (S_1 \oplus p_1, S_2 \odot p_2)$. We have that
 $I \models_O DL[\lambda; Q](\vec{t})$
 iff $O \cup \{S(\vec{e}) \mid p_1(\vec{e}) \in I\} \cup \{\neg S_2(\vec{e}) \mid p_2(\vec{e}) \notin I\} \models Q(\vec{t})$
 iff $O \cup \{S(\vec{e}) \mid p_1(\vec{e}) \in I\} \cup \{\neg S_2(\vec{e}) \mid \pi_{p_2}(\vec{e}) \in \pi_1(I)\} \models Q(\vec{t})$
 iff $O \cup \{S(\vec{e}) \mid p_1(\vec{e}) \in I \cup \pi_1(I)\} \cup \{\neg S_2(\vec{e}) \mid \pi_{p_2}(\vec{e}) \in I \cup \pi_1(I)\} \models Q(\vec{t})$
 iff $I \cup \pi_1(I) \models_O DL[S_1 \oplus p_1, S_2 \odot \pi_{p_2}; Q](\vec{t})$
 iff $I \cup \pi_1(I) \models_O DL[\pi(\lambda); Q](\vec{t})$
 iff $\pi(I) \not\models_O \pi(not A)$.

The above proof can be extended to the case where $\lambda = (S_1 \oplus p_1, \dots, S_m \oplus p_m, S'_1 \odot q_1, \dots, S'_n \odot q_n; Q](\vec{t})$. ■

Lemma 2 Let $\mathcal{K} = (O, P)$ be a dl-program and $I \subseteq HB_P$. Then

- (i) $\pi_1(I) = \{\pi_p(\vec{c}) \in HB_{\pi(P)}\} \cap lfp(\gamma_{[\pi(\mathcal{K})]^{s, \pi(I)}})$,
- (ii) $\pi_2(I) = \{\pi_A \in HB_{\pi(P)}\} \cap lfp(\gamma_{[\pi(\mathcal{K})]^{s, \pi(I)}})$, and
- (iii) $\gamma_{\mathcal{K}^{s, I}}^k = HB_P \cap \gamma_{[\pi(\mathcal{K})]^{s, \pi(I)}}^k$ for any $k \geq 0$.

Proof: (i) It is evident that, for any atom $\pi_p(\vec{c}) \in HB_{\pi(P)}$, the rule $(\pi_p(\vec{c}) \leftarrow not p(\vec{c}))$ is in $\pi(P)$. We have that

$\pi_p(\vec{c}) \in \pi_1(I)$
 iff $p(\vec{c}) \notin I$
 iff $p(\vec{c}) \notin \pi(I)$
 iff the rule $(\pi_p(\vec{c}) \leftarrow)$ belongs to $s[\pi(P)]_O^{s, \pi(I)}$
 iff $\pi_p(\vec{c}) \in lfp(\gamma_{[\pi(\mathcal{K})]^{s, \pi(I)}})$.

(ii) It is clear that, for any $\pi_A \in \pi_2(I)$, the rule $(\pi_A \leftarrow \pi(not A))$ is in $\pi(P)$ such that $A \in DL_P^?$ and $I \not\models_O A$. Let $A = DL[\lambda; Q](\vec{t})$. We have that

$\pi_A \in \pi_2(I)$
 iff $\pi_A \in HB_{\pi(P)}$ and $I \not\models_O A$
 iff $\pi(I) \not\models_O DL[\pi(\lambda); Q](\vec{t})$ (by (ii) of Lemma 1)
 iff the rule $(\pi_A \leftarrow)$ belongs to $s[\pi(P)]_O^{s, \pi(I)}$
 iff $\pi_A \in lfp(\gamma_{[\pi(\mathcal{K})]^{s, \pi(I)}})$.

(iii) We show this by induction on k .

Base: It is obvious for $k = 0$.

Step: Suppose it holds for $k = n$. Let us consider the case $k = n + 1$. For any atom $\alpha \in HB_P$, $\alpha \in \gamma_{\mathcal{K}^{s, I}}^{n+1}$ if and only if there is a rule

$$\alpha \leftarrow Pos, Ndl, not Neg$$

in P , where Pos is a set of atoms and monotonic dl-atoms and Ndl is a set of nonmonotonic dl-atoms such that

- $\gamma_{\mathcal{K}^s, I}^n \models_O A$ for any $A \in Pos$,
- $I \models_O B$ for any $B \in Ndl$, and
- $I \not\models_O C$ for any $C \in Neg$.

It follows that

- $\gamma_{\mathcal{K}^s, I}^n \models_O A$ if and only if $\gamma_{[\pi(\mathcal{K})]^s, \pi(I)}^n \models_O A$, by the inductive assumption,
- $I \models_O B$ if and only if $\pi_B \notin \pi(I)$, by the definition of $\pi_2(I)$, i.e., $\pi(I) \not\models_O \pi_B$, and
- $I \not\models_O C$ if and only if $\pi(I) \models_O \pi(not\ C)$ for any $C \in Neg$, by Lemma 1.

Thus we have that $\alpha \in \gamma_{\mathcal{K}^s, I}^{n+1}$ if and only if $\alpha \in \gamma_{[\pi(\mathcal{K})]^s, \pi(I)}^{n+1} \cap HB_P$. ■

Now we have the following key theorem: there exists a one-to-one mapping between the strong answer sets of a dl-program \mathcal{K} and those of $\pi(\mathcal{K})$.

Theorem 3 *Let $\mathcal{K} = (O, P)$ be a dl-program. Then*

- (i) *if I is a strong answer set of \mathcal{K} then $\pi(I)$ is a strong answer set of $\pi(\mathcal{K})$;*
- (ii) *if I^* is a strong answer set of $\pi(\mathcal{K})$ then $I^* \cap HB_P$ is a strong answer set of \mathcal{K} .*

Proof: (i) We have that

$$\begin{aligned}
 lfp(\gamma_{[\pi(\mathcal{K})]^s, \pi(I)}) &= lfp(\gamma_{[\pi(\mathcal{K})]^s, \pi(I)} \cap (HB_P \cup \{\pi_p(\vec{c}) \in HB_{\pi(P)}\} \cup \{\pi_A \in HB_{\pi(P)}\})) \\
 &= [HB_P \cap lfp(\gamma_{[\pi(\mathcal{K})]^s, \pi(I)})] \\
 &\quad \cup [\{\pi_p(\vec{c}) \in HB_{\pi(P)}\} \cap lfp(\gamma_{[\pi(\mathcal{K})]^s, \pi(I)})] \\
 &\quad \cup [\{\pi_A \in HB_{\pi(P)}\} \cap lfp(\gamma_{[\pi(\mathcal{K})]^s, \pi(I)})] \\
 &= [HB_P \cap \bigcup_{i \geq 0} \gamma_{[\pi(\mathcal{K})]^s, \pi(I)}^i] \cup \pi_1(I) \cup \pi_2(I), \text{ by (i) and (ii) of Lemma 2} \\
 &= \bigcup_{i \geq 0} [HB_P \cap \gamma_{[\pi(\mathcal{K})]^s, \pi(I)}^i] \cup \pi_1(I) \cup \pi_2(I) \\
 &= \bigcup_{i \geq 0} \gamma_{\mathcal{K}^s, I}^i \cup \pi_1(I) \cup \pi_2(I), \text{ by (iii) of Lemma 2} \\
 &= I \cup \pi_1(I) \cup \pi_2(I), \text{ since } I \text{ is a strong answer set of } \mathcal{K} \\
 &= \pi(I).
 \end{aligned}$$

It follows that $\pi(I)$ is a strong answer set of $\pi(\mathcal{K})$.

(ii) We prove $I^* = \pi(HB_P \cap I^*)$ first.

$$\begin{aligned}
 I^* &= I^* \cap (HB_P \cup \{\pi_p(\vec{c}) \in HB_{\pi(P)}\} \cup \{\pi_A \in HB_{\pi(P)}\}) \\
 &= (I^* \cap HB_P) \cup (I^* \cap \{\pi_p(\vec{c}) \in HB_{\pi(P)}\}) \cup (I^* \cap \{\pi_A \in HB_{\pi(P)}\}) \\
 &= (I^* \cap HB_P) \cup \pi_1(HB_P \cap I^*) \cup \pi_2(HB_P \cap I^*), \text{ by (i) and (ii) of Lemma 2} \\
 &= \pi(I^* \cap HB_P).
 \end{aligned}$$

Let $I = I^* \cap HB_P$. We have that

$$\begin{aligned}
 lfp(\gamma_{\mathcal{K}^s, I}) &= \bigcup_{i \geq 0} \gamma_{\mathcal{K}^s, I}^i \\
 &= \bigcup_{i \geq 0} (HB_P \cap \gamma_{[\pi(\mathcal{K})]^s, \pi(I)}^i), \text{ by (iii) of Lemma 2} \\
 &= HB_P \cap \bigcup_{i \geq 0} \gamma_{[\pi(\mathcal{K})]^s, \pi(I)}^i \\
 &= HB_P \cap lfp(\gamma_{[\pi(\mathcal{K})]^s, \pi(I)}) \\
 &= HB_P \cap \pi(I) \text{ since } \pi(I) = I^* \text{ is a strong answer set of } \pi(\mathcal{K}) \\
 &= I.
 \end{aligned}$$

It follows that I is a strong answer set of \mathcal{K} . ■

Please note that, we need to determine the monotonicity of dl-atoms in the translation π which is not tractable generally, and the translation does nothing for monotonic dl-atoms. That is, the “double negation” interpretation applies only to positive nonmonotonic dl-atoms. If we deviate from this condition, the translation no longer works for strong answer sets. For example, one may question whether monotonic dl-atoms can be handled like nonmonotonic dl-atoms, and if so, the translation turns out to be polynomial. Unfortunately we give a negative answer below.

Example 5 Consider the dl-program $\mathcal{K}_1 = (\emptyset, P_1)$ where $P_1 = \{p(a) \leftarrow DL[S \oplus p, S' \ominus q; S](a)\}$. The dl-atom $A = DL[S \oplus p, S' \ominus q; S](a)$ is monotonic. Thus, \mathcal{K}_1 is positive but neither canonical nor normal. It is evident that \emptyset is the unique strong answer set of \mathcal{K}_1 . If we apply π to eliminate the constraint operator in monotonic dl-atoms as what π does for nonmonotonic dl-atoms, we would get the dl-program (\emptyset, P'_1) where P'_1 consists of

$$p(a) \leftarrow not \pi_A, \quad \pi_A \leftarrow not DL[S \oplus p, S' \odot \pi_q; S](a), \quad \pi_q(a) \leftarrow not q(a).$$

One can verify that this dl-program has two strong answer sets, $\{p(a), \pi_q(a)\}$ and $\{\pi_A, \pi_q(a)\}$, which are $\{p(a)\}$ and \emptyset respectively when restricted to HB_P . However, we know that $\{p(a)\}$ is not a strong answer set of \mathcal{K}_1 . That is, such a translation may introduce some strong answer sets that do not correspond to any of the original dl-program in this case.

One may argue that π should treat monotonic dl-atoms in the same manner as treating nonmonotonic dl-atoms in default negation. However, for the dl-program $\mathcal{K}_2 = (\emptyset, P_2)$ where P_2 consists of

$$p(a) \leftarrow DL[S \odot p, S \ominus p; \neg S](a),$$

we have that the resulting dl-program (\emptyset, P'_2) where P'_2 consists of

$$p(a) \leftarrow DL[S \odot p, S \odot \pi_p, \neg S](a), \quad \pi_p(a) \leftarrow \text{not } p(a).$$

This dl-program has no strong answer sets at all. But the original dl-program has a unique strong answer $\{p(a)\}$. Even if we replace every p occurring in the dl-atom with π_p , the answer is still negative.

Similarly, we can show a one-to-one mapping between the weak answer sets of a dl-program \mathcal{K} and those of $\pi(\mathcal{K})$.

Theorem 4 *Let $\mathcal{K} = (O, P)$ be a dl-program. Then*

- (i) *if I is a weak answer set of \mathcal{K} , then $\pi(I)$ is a weak answer set of $\pi(\mathcal{K})$;*
- (ii) *if I^* is a weak answer set of $\pi(\mathcal{K})$, then $I^* \cap HB_P$ is a weak answer set of \mathcal{K} .*

Proof: See Appendix B. ■

As a matter of fact, there is a simpler translation that preserves weak answer sets of dl-programs.

Definition 2 ($\pi^*(\mathcal{K})$) *Let $\pi^*(\mathcal{K})$ be the same translation as $\pi(\mathcal{K})$ except that it does not distinguish nonmonotonic dl-atoms from dl-atoms, i.e., it handles monotonic dl-atoms in the way $\pi(\mathcal{K})$ deals with nonmonotonic dl-atoms.*

It is clear that π^* is polynomial. For instance, let us consider the dl-program \mathcal{K}_2 in Example 5. We have that $\pi^*(\mathcal{K}_2) = (\emptyset, \pi^*(P_2))$ where $\pi^*(P_2)$ consists of

$$p(a) \leftarrow \text{not } \pi_A, \quad \pi_p(a) \leftarrow \text{not } p(a), \quad \pi_A \leftarrow \text{not } DL[S \odot p, S \odot \pi_p, \neg S](a)$$

where $A = DL[S \odot p, S \odot \pi_p, \neg S](a)$. The interested readers can verify that $\{p(a)\}$ is the unique weak answer set of $\pi^*(\mathcal{K}_2)$.

Proposition 5 *Let $\mathcal{K} = (O, P)$ be a dl-program. Then*

- (i) *If $I \subseteq HB_P$ is a weak answer set of \mathcal{K} , then $\pi(I)$ is a weak answer set of $\pi^*(\mathcal{K})$.*
- (ii) *If I^* is a weak answer set of $\pi^*(\mathcal{K})$, then $I^* \cap HB_P$ is a weak answer set of \mathcal{K} .*

Proof: The proof is similar to the one of Theorem 4. ■

Note that, to remove the constraint operator from nonmonotonic dl-atoms of a dl-program, in general we must extend the underlying language. This is because there are dl-programs whose strong answer sets are not minimal, but the translated dl-program contains no nonmonotonic dl-atoms hence its strong answer sets are minimal (cf. Theorem 4.13 of (Eiter et al., 2008)). Therefore, we conclude that there is no transformation not using extra symbols that eliminates the constraint operator from normal dl-programs while preserving strong answer sets.

Recall that Motik and Rosati (2010) introduced a polynomial time transformation to translate a dl-atom mentioning no constraint operator into a first-order sentence and proved that, given a canonical dl-program \mathcal{K} , there is a one-to-one mapping between the strong answer sets of \mathcal{K} and the MKNF models of the corresponding MKNF knowledge base (Theorem 7.6 of (Motik & Rosati, 2010)). Theorem 3 above extends their result from canonical dl-programs to normal dl-programs, by applying the translation π first. In particular, the combined transformation is still polynomial for normal dl-programs.

4. Translating Dl-programs to Default Theories

Let us briefly recall the basic notions of default logic (Reiter, 1980). We assume a first-order language \mathcal{L} with a signature consisting of predicate, variable and constant symbols, including equality. A *default theory* Δ is a pair (D, W) where W is a set of closed formulas (sentences) of \mathcal{L} , and D is a set of *defaults* of the form:

$$\frac{\alpha : \beta_1, \dots, \beta_n}{\gamma} \quad (7)$$

where α (called *premise*), β_i ($0 \leq i \leq n$) (called *justification*), ⁸ γ (called *conclusion*) are formulas of \mathcal{L} . A default δ of the form (7) is *closed* if α, β_i ($1 \leq i \leq n$), γ are sentences, and a default theory is *closed* if all of its defaults are closed. In the following, we assume that every default theory is closed, unless stated otherwise. Let $\Delta = (D, W)$ be a default theory, and let S be a set of sentences. We define $\Gamma_\Delta(S)$ to be the smallest set satisfying

- $W \subseteq \Gamma_\Delta(S)$,
- $Th(\Gamma_\Delta(S)) = \Gamma_\Delta(S)$, and
- If δ is a default of the form (7) in D , and $\alpha \in \Gamma_\Delta(S)$, and $\neg\beta_i \notin S$ for each i ($1 \leq i \leq n$) then $\gamma \in \Gamma_\Delta(S)$,

where Th is the classical closure operator, i.e., $Th(\Sigma) = \{\psi \mid \Sigma \vdash \psi\}$ for a set of formulas Σ . A set of sentences E is an *extension* of Δ whenever $E = \Gamma_\Delta(E)$. Alternatively, a set of sentences E is an extension of Δ if and only if $E = \bigcup_{i \geq 0} E_i$, where

$$\begin{cases} E_0 = W, \\ E_{i+1} = Th(E_i) \cup \{\gamma \mid \frac{\alpha : \beta_1, \dots, \beta_n}{\gamma} \in D \text{ s.t. } \alpha \in E_i \text{ and } \neg\beta_1, \dots, \neg\beta_n \notin E\}, \quad i \geq 0. \end{cases} \quad (8)$$

It is not difficult to see that $\alpha \in E_i$ in (8) can be replaced by $E_i \vdash \alpha$.

In this section, we will present two approaches to translating a dl-program to a default theory which preserves the strong answer sets of dl-programs. In the first, if the given ontology is inconsistent, the resulting default theory is trivialized and possesses a unique extension that consists of all formulas of \mathcal{L} , while in the second, following the spirit of dl-programs, an inconsistent ontology does not trivialize the resulting default theory.⁹ Then we will give a translation from dl-programs under the weakly well-supported answer set semantics (Shen, 2011) to default theories. Before

8. Reiter (1980) used $n \geq 1$; the generalization we use is common and insignificant for our purposes.

9. The two approaches presented here do not preserve weak answer sets of dl-programs, for a good reason. Technically however, by applying a translation first that makes all dl-atoms occur negatively, we can obtain translations that preserve weak answer sets of dl-programs.

we proceed, let us comment on the impact of equality reasoning in the context of representing dl-programs by default logic.

4.1 Equality reasoning

The answer set semantics of dl-programs are defined with the intention that equality reasoning in the ontology is fully captured, while at the same time reasoning with rules is conducted relative to the Herbrand domain. The latter implies that equality reasoning is not carried over to reasoning with rules. For example, the dl-program

$$(O, P) = (\{a \approx b\}, \{p(a) \leftarrow \text{not } p(b), p(b) \leftarrow \text{not } p(a)\})$$

has two (strong) answer sets, $\{p(a)\}$ and $\{p(b)\}$, neither of which carries equality reasoning in the ontology to the rules. But if the dl-program is translated to the default theory $(\{\frac{\neg p(b)}{p(a)}, \frac{\neg p(a)}{p(b)}\}, \{a \approx b\})$ it has – evaluated under first-order logic with equality – no extensions. As suggested in (Eiter et al., 2008), one can emulate equality reasoning by imposing the unique name assumption (UNA) and a congruence relation on ontology.

Although congruence and UNA in general allow one to extend equality reasoning from the ontology to the rules, we will show that, for the purpose of representing dl-programs by default logic, for the standard default encoding like in the example above, strong answer sets are preserved by treating \approx as a congruence relation on ontology (i.e., replacement of equals by equals only applies to the predicates of the ontology); in particular, there is no need to adopt the UNA. For the default translation that handles inconsistent ontologies in the original spirit of dl-programs, neither congruence nor UNA is needed. These results provide additional insights in capturing dl-programs by default logic.

Thanks to Fitting, as shown by the following theorem, the equality \approx can be simulated by a congruence in the sense that a first-order formula with equality is satisfiable in a model with true equality if and only if it is satisfiable in a model where \approx is interpreted as a congruence relation.

Theorem 5 (Theorem 9.3.9 of (Fitting, 1996)) *Let \mathcal{L} be a first-order language, S a set of sentences and X a sentence. Then $S \models_{\approx} X$ iff $S \cup \text{eq}(\mathcal{L}) \models X$, where $S \models_{\approx} X$ means that X is true in every model of S in which \approx is interpreted as an equality relation and $\text{eq}(\mathcal{L})$ consists of the following axioms:*

$$\text{reflexivity} \quad (\forall x)(x \approx x), \quad (9)$$

$$\text{function replacement} \quad (\forall \vec{x}, \vec{y})[(\vec{x} \approx \vec{y}) \supset (f(\vec{x}) \approx f(\vec{y}))], \quad \text{for every function } f \text{ of } \mathcal{L}, \quad (10)$$

$$\text{predicate replacement} \quad (\forall \vec{x}, \vec{y})[(\vec{x} \approx \vec{y}) \supset (p(\vec{x}) \supset p(\vec{y}))], \quad \text{for every predicate } p \text{ of } \mathcal{L}. \quad (11)$$

Since \approx is a part of \mathcal{L} , the symmetry and transitivity of \approx in \mathcal{L} can be easily derived from (9) and (11) as illustrated by Fitting (1996). In what follows, we take \approx as a congruence, unless otherwise explicitly stated, and we write \models for \models_{\approx} when it is clear from its context,

Before giving the translation from dl-programs to default theories, we first present a transformation for dl-atoms, which will be referred to throughout this section. Let $\mathcal{K} = (O, P)$ be a dl-program (for convenience, assume O is already translated to a first-order theory), $I \in \text{HB}_P$ an interpretation, and $\tau(C)$ is a first-order sentence translated from C :

- if C is an atom in HB_P , then $\tau(C) = C$, and

- if C is a dl-atom of the form (1) then $\tau(C)$ is a first-order sentence

$$\left[\bigwedge_{1 \leq i \leq m} \tau(S_i \text{ op}_i p_i) \right] \supset Q(\vec{t}) \text{ , where}$$

$$\tau(S \text{ op } p) = \begin{cases} \bigwedge_{p(\vec{c}) \in HB_P} [p(\vec{c}) \supset S(\vec{c})] & \text{if } op = \oplus \\ \bigwedge_{p(\vec{c}) \in HB_P} [p(\vec{c}) \supset \neg S(\vec{c})] & \text{if } op = \odot \\ \bigwedge_{p(\vec{c}) \in HB_P} [\neg p(\vec{c}) \supset \neg S(\vec{c})] & \text{if } op = \ominus \end{cases}$$

where we identify $S(\vec{c})$ and $Q(\vec{t})$ with their corresponding first-order sentences respectively. Since \vec{t} and \vec{c} mention no variables, $\tau(C)$ has no free variables. Thus $\tau(C)$ is closed.

4.2 Translation trivializing inconsistent ontology knowledge bases

We present the first transformation from dl-programs to default theories which preserves strong answer sets of dl-programs without nonmonotonic dl-atoms.

Definition 3 ($\tau(\mathcal{K})$) *Let $\mathcal{K} = (O, P)$ be a dl-program. We define $\tau(\mathcal{K})$ to be the default theory $(\tau(P), \tau(O))$ as follows*

- $\tau(O)$ is the congruence rewriting of O , i.e., replacing true equality in O by a congruence; by abusing the symbol we denote the congruence by \approx , together with the axioms (9) and (11) for every predicate in the underlying language of O , denoted by \mathcal{A}_O .¹⁰ Given an ontology O , we assume the predicates in the underlying language of O are exactly the ones occurring in O .
- $\tau(P)$ consists of, for each dl-rule of the form (2) in P , the default

$$\frac{\bigwedge_{1 \leq i \leq m} \tau(B_i) : \neg \tau(B_{m+1}), \dots, \neg \tau(B_n)}{A}$$

where $\tau(C)$ is defined in the preceding subsection and equality \approx is now taken as the congruence relation above.

It is evident that, given a dl-program $\mathcal{K} = (O, P)$, every extension of $\tau(\mathcal{K})$ has the form $Th(I \cup \tau(O))$, for some $I \subseteq HB_P$. Thus, if O is consistent then every extension of $\tau(\mathcal{K})$ is consistent. On the other hand, if O is inconsistent then $\tau(\mathcal{K})$ has a unique extension which is inconsistent. It is clear that $\tau(\mathcal{K})$ is of polynomial size of the dl-program \mathcal{K} , since the size of HB_P is polynomial in the size of P .

Example 6 [Continued from Example 1]

- Note that $\tau(\mathcal{K}_1) = (\{d\}, W)$ where $W = \{\forall x. S(x) \supset S'(x)\} \cup \mathcal{A}_{O_1}$ and

$$d = \frac{(p(a) \supset S(a)) \supset S'(a)}{p(a)}.$$

It is easy to see that $\tau(\mathcal{K}_1)$ has a unique extension $Th(W)$.

10. Note that we do not need function replacement axioms here as there are no functions occurring in O .

- Note that $\tau(\mathcal{K}_2) = (\{d\}, W)$ where $W = \mathcal{A}_{O_2}$ and

$$d = \frac{(p(a) \supset S(a)) \wedge (q(a) \supset \neg S'(a)) \wedge (\neg q(a) \supset \neg S'(a)) \supset S(a) \wedge \neg S'(a)}{p(a)}.$$

One can verify that $Th(W)$ is the unique extension of $\tau(\mathcal{K}_2)$ though we know that \mathcal{K}_2 has two strong answer sets, \emptyset and $\{p(a)\}$.

The default theory $\tau(\mathcal{K}_2)$ in the above example shows that if a dl-program \mathcal{K} mentions non-monotonic dl-atoms, then $\tau(\mathcal{K})$ may have no corresponding extensions for some strong answer sets of \mathcal{K} . However, the one-to-one mapping between strong answer sets of \mathcal{K} and the extensions of $\tau(\mathcal{K})$ does exist for dl-programs mentioning no nonmonotonic dl-atoms and whose knowledge bases are consistent.

In the following, when it is clear from the context, we will identify a finite set S of formulas as the conjunction of elements in S for convenience. The following lemma relates a disjunctive normal form to a conjunctive normal form, which is well-known.

Lemma 3 *Let $A = \{A_1, \dots, A_n\}$, $B = \{B_1, \dots, B_n\}$ and $I = \{i \mid 1 \leq i \leq n\}$ where A_i, B_i ($1 \leq i \leq n$) are atoms. Then*

$$\bigvee_{I' \subseteq I} \left(\bigwedge_{i \in I'} A_i \wedge \bigwedge_{j \in I \setminus I'} B_j \right) \equiv \bigwedge_{i \in I} (A_i \vee B_i).$$

Lemma 4 *Let M be a set of ground atoms, ψ_i, φ_i and ϕ are formulas not mentioning true equality, the predicates p, p_1, p_2 and the predicates occurring in M , where $1 \leq i \leq n$. Then*

$$\begin{aligned} (1) \bigwedge_{1 \leq i \leq n} M \wedge \bigwedge_{1 \leq i \leq n} ((p(\vec{c}_i) \supset \psi_i) \wedge (\neg p(\vec{c}_i) \supset \varphi_i)) &\models \phi \text{ iff } \bigwedge_{p(\vec{c}_j) \in M} \psi_j \wedge \bigwedge_{p(\vec{c}_i) \notin M} (\psi_i \vee \varphi_i) \models \phi, \\ (2) \bigwedge_{1 \leq i \leq n} M \wedge \bigwedge_{1 \leq i \leq n} ((p_1(\vec{c}_i) \supset \psi_i) \wedge (\neg p_2(\vec{c}_i) \supset \varphi_i)) &\models \phi \text{ iff } \bigwedge_{p_1(\vec{c}_i) \in M} \psi_i \models \phi. \end{aligned}$$

Proof: (1) The direction from right to left is obvious as $(\alpha \supset \psi) \wedge (\neg \alpha \supset \varphi) \models \psi \vee \varphi$. Let us consider the other direction. It suffices to show

$$\bigwedge_{p(\vec{c}_i) \notin M} ((p(\vec{c}_i) \supset \psi_i) \wedge (\neg p(\vec{c}_i) \supset \varphi_i)) \models \phi \text{ only if } \bigwedge_{p(\vec{c}_i) \notin M} (\psi_i \vee \varphi_i) \models \phi. \quad (12)$$

Towards a contradiction, suppose that the left hand side of this statement holds and there is an interpretation $\mathcal{I} \models \bigwedge_{1 \leq i \leq n} (\psi_i \vee \varphi_i)$ and $\mathcal{I} \not\models \phi$. It follows that $\mathcal{I} \not\models \bigwedge_{1 \leq i \leq n} ((p(\vec{c}_i) \supset \psi_i) \wedge (\neg p(\vec{c}_i) \supset \varphi_i))$. Thus there exists some k ($1 \leq k \leq n$) such that $\mathcal{I} \not\models (p(\vec{c}_k) \supset \psi_k) \wedge (\neg p(\vec{c}_k) \supset \varphi_k)$. Without loss of generality, we assume $k = 1$. Let us consider the following two cases:

- $\mathcal{I} \models p(\vec{c}_1)$. In this case we have $\mathcal{I} \not\models \psi_1$, by which $\mathcal{I} \models \varphi_1$ due to $\mathcal{I} \models \psi_1 \vee \varphi_1$. As the formulas ψ_i, φ_i ($1 \leq i \leq n$) and ϕ do not involve the predicate p , the interpretation \mathcal{I}_1 which coincides with \mathcal{I} except that $\mathcal{I}_1 \not\models p(\vec{c}_1)$ satisfies the conditions $\mathcal{I}_1 \models \bigwedge_{1 \leq i \leq n} (\psi_i \vee \varphi_i)$ and $\mathcal{I}_1 \not\models \phi$. From $\mathcal{I} \models \varphi_1$ it follows that $\mathcal{I}_1 \models \varphi_1$; thus $\mathcal{I}_1 \models (p(\vec{c}_1) \supset \psi_1) \wedge (\neg p(\vec{c}_1) \supset \varphi_1)$. It follows that there exists some j ($2 \leq j \leq n$) such that $\mathcal{I}_1 \not\models (p(\vec{c}_j) \supset \psi_j) \wedge (\neg p(\vec{c}_j) \supset \varphi_j)$.

$(\neg p(\vec{c}_j) \supset \varphi_j)$. Without loss of generality, we can assume $j = 2$. With a similar case analysis and continuing the argument, it follows that there exists an interpretation \mathcal{I}_{n-1} such that $\mathcal{I}_{n-1} \models \bigwedge_{1 \leq i \leq n} (\psi_i \vee \varphi_i)$, $\mathcal{I}_{n-1} \not\models \phi$ and $\mathcal{I}_{n-1} \models \bigwedge_{1 \leq i \leq n-1} ((p(\vec{c}_i) \supset \psi_i) \wedge (\neg p(\vec{c}_i) \supset \varphi_i))$. It follows that $\mathcal{I}_{n-1} \not\models (p(\vec{c}_n) \supset \psi_n) \wedge (\neg p(\vec{c}_n) \supset \varphi_n)$. We can finally construct an interpretation \mathcal{I}_n in a similar way that satisfies

- $\mathcal{I}_n \models \bigwedge_{1 \leq i \leq n} (\psi_i \vee \varphi_i)$,
- $\mathcal{I}_n \not\models \phi$, and
- $\mathcal{I}_n \models \bigwedge_{1 \leq i \leq n} ((p(\vec{c}_i) \supset \psi_i) \wedge (\neg p(\vec{c}_i) \supset \varphi_i))$.

As the latter combined with the assumption implies $\mathcal{I}_n \models \phi$, we have a contradiction.

- $\mathcal{I} \not\models p(\vec{c}_1)$. Similar to the previous case.

(2) The direction from right to left is obvious again. For the other direction, suppose that there is an interpretation \mathcal{I} such that $\mathcal{I} \models \bigwedge_{p_1(\vec{c}_i) \in M} \psi_i$ and $\mathcal{I} \not\models \phi$. We construct an interpretation \mathcal{I}' , which is the same as \mathcal{I} except that $\mathcal{I}' \models \bigwedge M$, $\mathcal{I}' \not\models p_1(c_i)$ if $p_1(c_i) \notin M$, and $\mathcal{I}' \models p_2(\vec{c}_j)$ if $p_2(c_j) \notin M$, for every $1 \leq i, j \leq n$. It is clear that $\mathcal{I}' \models \bigwedge_{p_1(\vec{c}_i) \in M} \psi_i$ and $\mathcal{I}' \not\models \phi$. However, we have $\mathcal{I}' \models \bigwedge M \wedge \bigwedge_{1 \leq i \leq n} ((p_1(\vec{c}_i) \supset \psi_i) \wedge (\neg p_2(\vec{c}_i) \supset \varphi_i))$, which implies $\mathcal{I}' \models \phi$, a contradiction. ■

Please note here that, in the above lemma, it is crucial that ψ_i, φ_i and ϕ mention no true equality. Otherwise, one can check that, if \approx is taken as true equality, then on the one hand we have

$$[(p(c_1) \supset c_1 \approx c_2) \wedge (\neg p(c_1) \supset q)] \wedge [(p(c_2) \supset q) \wedge (\neg p(c_2) \supset \neg q)] \models q$$

and on the other we have $(c_1 \approx c_2 \vee q) \not\models q$. It is clear that this discrepancy will not arise if \approx is treated as a congruence relation and there is no predicate replacement axiom for the predicate p .

Lemma 5 *Let $\mathcal{K} = (O, P)$ be a dl-program and $I \subseteq HB_P$. Then*

- (i) *If A is an atom in HB_P and O is consistent, then $I \models_O A$ iff $\tau(O) \cup I \models \tau(A)$.*
- (ii) *If $A = DL[\lambda; Q](\vec{t})$ is a monotonic dl-atom, then $I \models_O A$ iff $\tau(O) \cup I \models \tau(A)$.*

Proof: (i) Since A is an atom and O mentions no predicates occurring in I , we have that $\tau(O) \cup I$ is consistent if and only if O is consistent. It follows that $I \models_O A$ iff $A \in I$ iff $I \models \tau(A)$ since $\tau(A) = A$. It is obvious that if $I \models \tau(A)$ then $\tau(O) \cup I \models \tau(A)$. It remains to show that $I \models \tau(A)$ if $\tau(O) \cup I \models \tau(A)$. Suppose $I \not\models \tau(A)$, i.e., $\tau(A) \notin I$. Thus there exists an interpretation \mathcal{I} such that $\mathcal{I} \models I$ and $\mathcal{I} \not\models \tau(A)$. Recall that $\tau(O)$ has no equality, and it has no predicates in common with I . We can construct an interpretation \mathcal{I}^* which coincides with \mathcal{I} except that $\mathcal{I}^* \models \tau(O)$. It follows $\mathcal{I}^* \models \tau(A)$ by $\mathcal{I}^* \models \tau(O) \cup I$, which contradicts $\mathcal{I} \not\models \tau(A)$ as \mathcal{I} coincides with \mathcal{I}^* for the predicate occurring in $\tau(A)$.

(ii) For clarity, and without loss of generality, let $\lambda = (S_1 \oplus p_1, S_2 \ominus p_2)$. We have that

$$\begin{aligned} \tau(O) \cup I &\models \tau(A) \text{ iff} \\ \tau(O) \cup I &\models \left(\bigwedge_{p_1(\vec{e}) \in HB_P} (p_1(\vec{e}) \supset S_1(\vec{e})) \wedge \bigwedge_{p_2(\vec{e}) \in HB_P} (\neg p_2(\vec{e}) \supset \neg S_2(\vec{e})) \right) \supset Q(\vec{t}) \text{ iff} \\ I \wedge \bigwedge_{p_1(\vec{e}) \in HB_P} (p_1(\vec{e}) \supset S_1(\vec{e})) \wedge \bigwedge_{p_2(\vec{e}) \in HB_P} (\neg p_2(\vec{e}) \supset \neg S_2(\vec{e})) &\models \tau(O) \supset Q(\vec{t}). \end{aligned} \quad (13)$$

Let us consider the following two cases:

(a) $p_1 \neq p_2$. We have that Equation (13) holds iff $\{S_1(\vec{e}) \mid p_1(\vec{e}) \in I\} \models \tau(O) \supset Q(\vec{t})$ by (2) of Lemma 4. It follows that

$$\begin{aligned} &\{S_1(\vec{e}) \mid p_1(\vec{e}) \in I\} \models \tau(O) \supset Q(\vec{t}) \\ \Rightarrow &\{S_1(\vec{e}) \mid p_1(\vec{e}) \in I\} \cup \{\neg S_2(\vec{e}) \mid p_2(\vec{e}) \notin I\} \models \tau(O) \supset Q(\vec{t}) \\ \Rightarrow &\tau(O) \cup \{S_1(\vec{e}) \mid p_1(\vec{e}) \in I\} \cup \{\neg S_2(\vec{e}) \mid p_2(\vec{e}) \notin I\} \models Q(\vec{t}) \\ \Rightarrow &O \cup \{S_1(\vec{e}) \mid p_1(\vec{e}) \in I\} \cup \{\neg S_2(\vec{e}) \mid p_2(\vec{e}) \notin I\} \models Q(\vec{t}) \text{ (now } \approx \text{ is taken as an equality, by Theorem 5)} \\ \Rightarrow &I \models_O A. \end{aligned}$$

On the other hand, let $I' = \{p_2(\vec{e}) \in HB_P\}$. We have that

$$\begin{aligned} &I \models_O A \\ \Rightarrow &I \cup I' \models_O A \text{ (since } A \text{ is monotonic)} \\ \Rightarrow &O \cup \{S_1(\vec{e}) \mid p_1(\vec{e}) \in I \cup I'\} \cup \{\neg S_2(\vec{e}) \mid p_2(\vec{e}) \notin I \cup I'\} \models Q(\vec{t}) \\ \Rightarrow &O \cup \{S_1(\vec{e}) \mid p_1(\vec{e}) \in I \cup I'\} \models Q(\vec{t}) \\ \Rightarrow &O \cup \{S_1(\vec{e}) \mid p_1(\vec{e}) \in I\} \models Q(\vec{t}) \\ \Rightarrow &\{S_1(\vec{e}) \mid p_1(\vec{e}) \in I\} \models O \supset Q(\vec{t}) \\ \Rightarrow &\{S_1(\vec{e}) \mid p_1(\vec{e}) \in I\} \models \tau(O) \supset Q(\vec{t}) \text{ (now } \approx \text{ is taken as a congruence, by Theorem 5).} \end{aligned}$$

(b) $p_1 = p_2 = p$. By (1) of Lemma 4, we have that Equation (13) holds iff

$$\{S_1(\vec{e}) \mid p(\vec{e}) \in I\} \cup \{S_1(\vec{e}) \vee \neg S_2(\vec{e}) \mid p(\vec{e}) \in HB_P \setminus I\} \models \tau(O) \supset Q(\vec{t}).$$

It follows that

$$\begin{aligned} &\{S_1(\vec{e}) \mid p(\vec{e}) \in I\} \cup \{S_1(\vec{e}) \vee \neg S_2(\vec{e}) \mid p(\vec{e}) \in HB_P \setminus I\} \models \tau(O) \supset Q(\vec{t}) \\ \Rightarrow &\{S_1(\vec{e}) \mid p(\vec{e}) \in I\} \cup \{\neg S_2(\vec{e}) \mid p(\vec{e}) \in HB_P \setminus I\} \models \tau(O) \supset Q(\vec{t}) \\ \Rightarrow &\tau(O) \cup \{S_1(\vec{e}) \mid p(\vec{e}) \in I\} \cup \{\neg S_2(\vec{e}) \mid p(\vec{e}) \notin I\} \models Q(\vec{t}) \\ \Rightarrow &O \cup \{S_1(\vec{e}) \mid p(\vec{e}) \in I\} \cup \{\neg S_2(\vec{e}) \mid p(\vec{e}) \notin I\} \models Q(\vec{t}) \text{ (now } \approx \text{ is taken as an equality, by Theorem 5)} \\ \Rightarrow &I \models_O A. \end{aligned}$$

Conversely, suppose $I \models_O A$. Let $M_1 = \{S_1(\vec{e}) \mid p(\vec{e}) \in HB_P \setminus I\} = \{S_1(\vec{e}_i) \mid 1 \leq i \leq k\}$, $M_2 = \{\neg S_2(\vec{e}) \mid p(\vec{e}) \in HB_P \setminus I\} = \{\neg S_2(\vec{e}_i) \mid 1 \leq i \leq k\}$ and $J = \{i \mid 1 \leq i \leq k\}$. Since A is monotonic, for any $J' \subseteq J$, we have that $I \cup \{p(\vec{e}_i) \mid i \in J'\} \models_O A$, i.e.,

$$\{S_1(\vec{e}) \mid p(\vec{e}) \in I\} \cup \{S_1(\vec{e}_i) \mid i \in J'\} \cup \{\neg S_2(\vec{e}_i) \mid i \in J \setminus J'\} \models O \supset Q(\vec{t}).$$

It follows that

$$\bigwedge_{p(\vec{e}) \in I} S_1(\vec{e}) \wedge \bigvee_{J' \subseteq J} \left(\bigwedge_{i \in J'} S_1(\vec{e}_i) \wedge \bigwedge_{i \in J \setminus J'} \neg S_2(\vec{e}_i) \right) \models O \supset Q(\vec{t})$$

which implies, by Lemma 3,

$$\bigwedge_{p(\vec{e}) \in I} S_1(\vec{e}) \wedge \bigwedge_{i \in J} (S_1(\vec{e}_i) \vee \neg S_2(\vec{e}_i)) \models O \supset Q(\vec{t})$$

i.e.,

$$\bigwedge_{p(\vec{e}) \in I} S_1(\vec{e}) \wedge \bigwedge_{p(\vec{e}) \in HB_P \setminus I} (S_1(\vec{e}) \vee \neg S_2(\vec{e})) \models O \supset Q(\vec{t}),$$

and equivalently

$$\bigwedge_{p(\vec{e}) \in I} S_1(\vec{e}) \wedge \bigwedge_{p(\vec{e}) \in HB_P \setminus I} (S_1(\vec{e}) \vee \neg S_2(\vec{e})) \models \tau(O) \supset Q(\vec{t}),$$

where \approx is taken as a congruence relation. Consequently, $I \models_O A$ iff $\tau(O) \cup I \models \tau(A)$. \blacksquare

We note that, in (i) of the above lemma, we can not replace “ $\tau(O) \cup I \models \tau(A)$ ” by “ $O \cup I \models \tau(A)$ ” since $O \cup I \models A$ does not imply $\tau(O) \cup I \models A$. For instance, let $O = \{a \approx b\}$, $I = \{p(a)\}$ and $A = p(b)$ where p is a predicate not belonging to the ontology and \approx is equality. Then we have that $\{a \approx b\} \cup \{p(a)\} \models p(b)$ as \approx is an equality, but $\tau(O) \cup \{p(a)\} \not\models p(b)$ as $\tau(O) = \{a \approx b\}$ with \approx being a congruence relation; as p does not occur in O , no replacement axiom of p is in $\tau(O)$.

Lemma 6 *Let $\mathcal{K} = (O, P)$ be a dl-program and $I \subseteq HB_P$ where O is consistent and $DL_P^? = \emptyset$. Then $\gamma_{\mathcal{K}^s, I}^i = E_i \cap HB_P$ for any $i \geq 0$, where E_i is defined as (8) for $\tau(\mathcal{K})$ and $E = Th(\tau(O) \cup I)$.*

Proof: We prove this by induction on i .

Base: If $i = 0$ then it is obvious since $\tau(O)$ is consistent (as O is consistent) and $E_0 = \tau(O)$.

Step: Suppose it holds for $i = n$. Now for any $h \in HB_P$, $h \in \gamma_{\mathcal{K}^s, I}^{n+1}$ if and only if there exists a dl-rule $(h \leftarrow Pos, not Neg)$ in P such that

- $\gamma_{\mathcal{K}^s, I}^n \models_O A$ for any $A \in Pos$, and
- $I \not\models_O B$ for any $B \in Neg$.

We have that

(i) $I \not\models_O B$

iff $\tau(O) \cup I \not\models \tau(B)$ (by Lemma 5 and $DL_P^? = \emptyset$)

iff $E \not\models \tau(B)$.

(ii) $\gamma_{\mathcal{K}^s, I}^n \models_O A$

iff $E_n \cap HB_P \models_O A$ (by inductive assumption)

iff $\tau(O) \cup E_n \cap HB_P \models \tau(A)$ (by Lemma 5 and $DL_P^? = \emptyset$)

iff $E_n \models \tau(A)$ (since $\tau(O) \subseteq E_n \subseteq Th(\tau(O) \cup HB_P)$).

Consequently we have $\gamma_{\mathcal{K}^s, I}^i = E_i \cap HB_P$ for any $i \geq 0$. \blacksquare

Theorem 6 *Let $\mathcal{K} = (O, P)$ be a dl-program such that $DL_P^? = \emptyset$ and $I \subseteq HB_P$. If O is consistent then I is a strong answer set of \mathcal{K} if and only if $E = Th(\tau(O) \cup I)$ is an extension of $\tau(\mathcal{K})$.*

Proof: (\Rightarrow) It suffices to show $E = \bigcup_{i \geq 0} E_i$ where E_i is defined as (8) for $\tau(\mathcal{K})$ and E .

$$\begin{aligned}
& E = Th(\tau(O) \cup I) \\
& \Rightarrow E \equiv \tau(O) \cup \gamma_{\mathcal{K}^s, I}^\infty \text{ (since } I = \gamma_{\mathcal{K}^s, I}^\infty) \\
& \Rightarrow E \equiv \tau(O) \cup \bigcup_{i \geq 0} E_i \cap HB_P \text{ (by Lemma 6)} \\
& \Rightarrow E \equiv \bigcup_{i \geq 0} E_i \cap HB_P \cup \tau(O) \\
& \Rightarrow E \equiv \bigcup_{i \geq 0} E_i \text{ (since } \tau(O) \subseteq E_i \subseteq Th(\tau(O) \cup HB_P)) \\
& \Rightarrow E = \bigcup_{i \geq 0} E_i \\
& \Rightarrow E \text{ is an extension of } \tau(\mathcal{K}). \\
& \quad (\Leftarrow) E \text{ is an extension of } \tau(\mathcal{K}) \\
& \Rightarrow E = \bigcup_{i \geq 0} E_i \text{ where } E_i \text{ is defined as (8) for } \tau(\mathcal{K}) \text{ and } E \\
& \Rightarrow Th(\tau(O) \cup I) = \bigcup_{i \geq 0} E_i \\
& \Rightarrow Th(\tau(O) \cup I) \cap HB_P = \left(\bigcup_{i \geq 0} E_i \right) \cap HB_P \\
& \Rightarrow I = \bigcup_{i \geq 0} (E_i \cap HB_P) \\
& \Rightarrow I = \gamma_{\mathcal{K}^s, I}^\infty \text{ (by Lemma 6)} \\
& \Rightarrow I = lfp(\gamma_{\mathcal{K}^s, I}) \\
& \Rightarrow I \text{ is a strong answer set of } \mathcal{K}. \quad \blacksquare
\end{aligned}$$

Since dl-programs can be translated into ones without nonmonotonic dl-atoms according to Theorem 3, we immediately have the following:

Corollary 7 *Let $\mathcal{K} = (O, P)$ be a dl-program and $I \subseteq HB_P$. If O is consistent then I is a strong answer set of \mathcal{K} if and only if $Th(\tau(O) \cup \pi(I))$ is an extension of $\tau(\pi(\mathcal{K}))$.*

Proof: I is a strong answer set of \mathcal{K}
iff $\pi(I)$ is a strong answer set of $\pi(\mathcal{K})$ (by Theorem 3)
iff $Th(\tau(O) \cup \pi(I))$ is an extension of $\tau(\pi(\mathcal{K}))$ (by Theorem 6). \blacksquare

Although the translation τ given here is kind of “standard”, as it draw ideas from (Gelfond & Lifschitz, 1991) and (Motik & Rosati, 2010), there are a number of subtleties in dealing with dl-programs which make it non-trivial, in addition to the problem of equality.

In translating dl-programs to MKNF knowledge bases, Motik and Rosati (2010) did not consider dl-atoms containing the constraint operator. In addition, there is an essential difference in that their approach does not work here as illustrated by the next example.

Example 7 Let $\mathcal{K} = (O, P)$ where $O = \{S(b)\}$, b an individual in the description logic but not a constant occurring in P , and P consist of

$$p(a) \leftarrow DL[S \ominus p, S \odot p; S](a).$$

It is trivial that $HB_P = \{p(a)\}$ and there is no interpretation of \mathcal{K} satisfying the dl-atom $DL[S \ominus p, S \odot p; S](a)$, thus it is monotonic and then the unique strong answer set of \mathcal{K} is \emptyset . In terms of Motik and Rosati’s translation, we would have the default theory $\Delta = (\{d\}, O)$ where

$$d = \frac{(\forall x.(p(x) \supset \neg S(x)) \wedge \forall x.(\neg p(x) \supset \neg S(x))) \rightarrow S(a)}{p(a)}.$$

Since the sentence $\forall x.(p(x) \supset \neg S(x)) \wedge \forall x.(\neg p(x) \supset \neg S(x))$ is classically equivalent to $\forall x.\neg S(x)$, the unique extension of Δ is $Th(\{S(b), p(a)\})$; when restricted to HB_P , it is $\{p(a)\}$ which corresponds to no answer set of \mathcal{K} at all. It is not difficult to check that the default theory $\tau(\mathcal{K})$ has a unique extension $Th(\{S(b)\})$ which corresponds to the strong answer set \emptyset of \mathcal{K} .

Another subtle point is that the default translation alone may not capture the semantics of a dl-program. If a dl-program \mathcal{K} mentions nonmonotonic dl-atoms then it is possible that $\tau(\mathcal{K})$ has some extensions that do not correspond to any strong answer sets of \mathcal{K} .

Example 8 Let $\mathcal{K} = (O, P)$ where $O = \emptyset$ and P consists of

$$\begin{aligned} p(a) &\leftarrow q(a), \\ q(a) &\leftarrow DL[S_1 \oplus p, S_2 \ominus q; S_1 \sqcup \neg S_2](a). \end{aligned}$$

It is not difficult to check that $A = DL[S_1 \oplus p, S_2 \ominus q; S_1 \sqcup \neg S_2](a)$ is nonmonotonic and \mathcal{K} has a unique strong answer set $\{p(a), q(a)\}$. But note that the default theory $\tau(\mathcal{K}) = (D, W)$ where $W = \mathcal{A}_O$ and D consists of

$$\frac{q(a) :}{p(a)}, \quad \frac{(p(a) \supset S_1(a)) \wedge (\neg q(a) \supset \neg S_2(a)) \supset (S_1(a) \vee \neg S_2(a)) :}{q(a)}$$

has a unique extension $Th(W)$ which does not correspond to any strong answer set of \mathcal{K} . However, if we apply the translation π to \mathcal{K} first, we will have the dl-program $\pi(\mathcal{K}) = (O, \pi(P))$, where $\pi(P)$ consists of

$$\begin{aligned} p(a) &\leftarrow q(a), & q(a) &\leftarrow not \pi_A, & \pi_q(a) &\leftarrow not q(a), \\ \pi_A &\leftarrow not DL[S_1 \oplus p, S_2 \odot \pi_q; S_1 \sqcup \neg S_2](a). \end{aligned}$$

It is tedious but not difficult to check that the unique strong answer set of $\pi(\mathcal{K})$ is $\{p(a), q(a)\}$. When we apply the translation τ to $\pi(\mathcal{K})$, we have the default theory $\tau(\pi(\mathcal{K})) = (D', W')$ where $W' = \mathcal{A}_O$ and D' consists of

$$\frac{\frac{q(a) :}{p(a)}, \quad \frac{: \neg \pi_A}{q(a)}, \quad \frac{: \neg q(a)}{\pi_q(a)}}{: \neg [(p(a) \supset S_1(a)) \wedge (\pi_q(a) \supset \neg S_2(a)) \supset (S_1(a) \vee \neg S_2(a))]} \pi_A$$

The interested reader can verify that the unique extension of $\tau(\pi(\mathcal{K}))$ is $Th(\tau(O) \cup \{p(a), q(a)\})$, which corresponds to the unique strong answer set of \mathcal{K} .

We note that the translation τ does not preserve weak answer sets of a normal dl-program, as shown by $\tau(\mathcal{K}_2)$ in Example 6, not even for canonical dl-programs, as shown by $\tau(\mathcal{K}_1)$ in Example 6.

To preserve the weak answer sets of a dl-program, one may attempt to “shift” $\tau(\cdot)$ from premise to justification of a default in the translation τ ; however, this does not work. Consider the dl-program $\mathcal{K} = (\emptyset, P)$ where $P = \{p(a) \leftarrow DL[S \oplus p, S](a)\}$. Under the suggestion, we would have obtained the default theory $\Delta = (D, W)$, where $W = \tau(\emptyset)$ and D consists of

$$\frac{: (p(s) \supset S(a)) \supset S(a)}{p(a)}.$$

It is clear that Δ has a unique extension $Th(\tau(\emptyset) \cup \{p(a)\})$, but we know that \mathcal{K} has two weak answer sets, \emptyset and $\{p(a)\}$. This issue can be addressed by a translation which makes all dl-atoms occur negatively.

Definition 4 ($\sigma(\mathcal{K})$) *Let r be a dl-rule of the form (2). We define $\sigma(r)$ to be the rule*

$$A \leftarrow not \sigma(B_1), \dots, not \sigma(B_m), \dots, not B_{m+1}, \dots, not B_n$$

where $\sigma(B) = \sigma_B$ if B is a dl-atom, and B otherwise, where σ_B is a fresh propositional atom. For every dl-program $\mathcal{K} = (O, P)$, we define $\sigma(\mathcal{K}) = (O, \sigma(P))$ where $\sigma(P)$ consists of the rules in

$$\{\sigma(r) \mid r \in P\} \cup \{\sigma_B \leftarrow not B \mid B \in DL_P\}.$$

Example 9 Let us consider the above dl-program $\mathcal{K} = (O, P)$ where $O = \emptyset$ and $P = \{p(a) \leftarrow DL[S \oplus p, S](a)\}$. We have that $\sigma(\mathcal{K}) = (O, \sigma(P))$ where $A = DL[S \oplus p, S](a)$ and $\sigma(P)$ consists of the below two dl-rules:

$$p(a) \leftarrow not \sigma_A, \quad \sigma_A \leftarrow not DL[S \oplus p, S](a).$$

It is easy to see that $\sigma(\mathcal{K})$ has two weak answer sets $\{\sigma_A\}$ and $\{p(a)\}$.

Proposition 6 *Let $\mathcal{K} = (O, P)$ be a dl-program and $I \subseteq HB_P$. Then I is a weak answer set of \mathcal{K} iff I' is a weak answer set of $\sigma(\mathcal{K})$ where $I' = I \cup \{\sigma_B \mid B \in DL_P \text{ and } I \not\models_O B\}$.*

Proof: As $\sigma_B \in I'$ iff $I \not\models_O B$ for any $B \in DL_P$, we have that $wP_O^I \subseteq w[\sigma(P)]_O^{I'}$ and for any rule $(h \leftarrow Pos)$ in $w[\sigma(P)]_O^{I'} \setminus wP_O^I$, $Pos = \emptyset$ and h has the form σ_B for some $B \in DL_P$. Thus we have $I' \setminus I = lfp(\gamma_{[\sigma(\mathcal{K})]^{w, I'}}) \cap \{\sigma_B \mid B \in DL_P\}$ and $lfp(\gamma_{[\mathcal{K}^{w, I}]} \cup (I \setminus I')) = lfp(\gamma_{[\sigma(\mathcal{K})]^{w, I'}})$. This completes the proof. ■

Proposition 7 *Let $\mathcal{K} = (O, P)$ be a dl-program such that O is consistent, $DL_P^2 = \emptyset$ and all dl-atoms occur negatively in P , i.e., for any rule $(h \leftarrow Pos, not Neg)$ of P , there is no dl-atom in Pos . Then an interpretation $I \subseteq HB_P$ is a weak answer set of \mathcal{K} iff $E = Th(I \cup \tau(O))$ is an extension of $\tau(\mathcal{K})$.*

Proof: By Lemma 5, we can inductively prove $\gamma_{[\mathcal{K}^{w, I}]}^i = E_i \cap HB_P$ for any $i \geq 0$ where E_i is defined as (8) for E and $\tau(\mathcal{K})$. The remainder of the proof is similar to the one of Theorem 6. ■

Together with Theorem 4, the above two propositions imply a translation from dl-programs with consistent ontologies under the weak answer set semantics to default theories.

Corollary 8 *Let $\mathcal{K} = (O, P)$ be a dl-program where O is consistent. The below conditions are equivalent to each other:*

- (i) *An interpretation $I \subseteq HB_P$ is a weak answer set \mathcal{K} .*
- (ii) *$Th(\tau(O) \cup \pi(I'))$ is an extension of $\tau(\pi(\sigma(\mathcal{K})))$ where $I' = I \cup \{\sigma_B \mid B \in DL_P \text{ and } I \not\models_O B\}$.*

One can easily see that the translation $\sigma \cdot \pi \cdot \tau$, i.e., applying σ firstly then π and finally τ , is polynomial. Thus, under the weak answer set semantics, we obtain a polynomial, faithful and modular translation from dl-programs with consistent ontologies to default theories.

4.3 Handling inconsistent ontology knowledge bases

A dl-program may have nontrivial strong answer sets even if its ontology knowledge base is inconsistent. For instance, let $\mathcal{K} = (O, P)$, where $O = \{S(a), \neg S(a)\}$ and $P = \{p \leftarrow \text{not } q, q \leftarrow \text{not } p\}$. Obviously \mathcal{K} has two strong answer sets, $\{p\}$ and $\{q\}$, while the translation introduced in the last subsection, $\tau(\mathcal{K})$, yields a unique extension which is inconsistent. In combining different knowledge bases, it is highly desirable that the whole system is not trivialized due to the imperfection of a subsystem. For dl-programs, this feature is naturally built into the strong answer set semantics. When considering embedding, it is important that this feature be preserved.

In Theorem 6 and Corollary 7, we require O to be consistent and we assume a limited congruence rewriting, i.e., the equality \approx is understood as a congruence and the congruence is applied only to the predicates of underlying description logic. To relax these conditions, we propose the following translation τ' which is slightly different from τ .

Definition 5 *Given a dl-program $\mathcal{K} = (O, P)$, $\tau'(\mathcal{K})$ is the default theory (D, \emptyset) , where D is the same as the one in the definition of τ except for dl-atoms. Suppose A is a dl-atom of the form (1). We define $\tau'(A)$ to be the first-order sentence:*

$$\left[O \wedge \left(\bigwedge_{1 \leq i \leq m} \tau(S_i \text{ op}_i p_i) \right) \right] \supset Q(\vec{t})$$

where O is identified with its corresponding first-order theory in which we do not require equality to be a congruence.

Evidently, given a dl-program \mathcal{K} , every extension of $\tau'(\mathcal{K})$ is consistent and has the form $Th(I)$ for some $I \subseteq HB_P$.

Example 10 Let $\mathcal{K} = (O, P)$ be a dl-program where $O = \{S(a), \neg S'(a), S \sqsubseteq S'\}$ and P consists of $p(a) \leftarrow DL[S \oplus p; \neg S](a)$. It is evident that O is inconsistent and \mathcal{K} has a unique strong answer set $\{p(a)\}$. Now we have that the corresponding first-order theory of O is $S(a) \wedge \neg S'(a) \wedge (\forall x. S(x) \supset S'(x))$, and $\tau'(\mathcal{K}) = (\{d\}, \emptyset)$ where

$$d = \frac{(O \wedge (p(a) \supset S(a))) \supset \neg S(a)}{p(a)}.$$

It is not difficult to verify that $E = Th(\{p(a)\})$ is the unique extension of $\tau'(\mathcal{K})$ which is consistent, while the unique extension of $\tau(\mathcal{K})$ is inconsistent.

Different from τ in another aspect, the translation τ' keeps equality as equality. For instance, **for** the dl-program \mathcal{K} in Section 4.1, we have that $\tau'(\mathcal{K}) = (D, \emptyset)$ where $D = \left\{ \frac{\neg p(a)}{p(b)}, \frac{\neg p(b)}{p(a)} \right\}$. Evidently, the default theory $\tau'(\mathcal{K})$ has two extensions $Th(\{p(a)\})$ and $Th(\{p(b)\})$.

The translation τ' is obviously modular. We will show below that it is faithful.

Lemma 7 *Let $\mathcal{K} = (O, P)$ be a dl-program, A an atom or a monotonic dl-atom and $I \subseteq HB_P$. Then $I \models_O A$ if and only if $I \vdash \tau'(A)$.*

Proof: The conclusion is evident if A is an atom or O is inconsistent. Suppose A is a dl-atom and O is consistent. Let $A = DL[\lambda; Q](\vec{t})$. Thus $\tau(A)$ is of the form $\psi \supset Q(\vec{t})$ which implies $\tau'(A) \equiv (O \wedge \psi) \supset Q(\vec{t})$. We have that

$$I \models_O A$$

iff $\tau(O) \cup I \vdash \tau(A)$ (by Lemma (ii) of 5, where \approx is taken as a congruence relation)

$$\text{iff } I \vdash \tau(O) \supset \tau(A)$$

$$\text{iff } I \vdash \tau(O) \supset (\psi \supset Q(\vec{t}))$$

$$\text{iff } I \vdash (\tau(O) \wedge \psi) \supset Q(\vec{t})$$

iff $I \vdash (O \wedge \psi) \supset Q(\vec{t})$ (by Theorem 5, where \approx is taken as equality)

$$\text{iff } I \vdash \tau'(A).$$

■

Lemma 8 Let $\mathcal{K} = (O, P)$ be a dl-program such that $DL_P^? = \emptyset$, $I \subseteq HB_P$ and $E = Th(I)$. Then $\gamma_{\mathcal{K}^s, I}^i = E_i \cap HB_P$ for any $i \geq 0$, where E_k is defined as (8) for $\tau'(\mathcal{K})$ and E .

Proof: We prove this by induction on k .

Base: It is obvious for $i = 0$ since $E_0 = \emptyset$.

Step: Suppose it holds for $i = n$. For any $h \in HB_P$, $h \in \gamma_{\mathcal{K}^s, I}^{n+1}$ if and only if there exists a dl-rule $(h \leftarrow Pos, not Neg)$ such that

- $\gamma_{\mathcal{K}^s, I}^n \models_O A$ for any $A \in Pos$, and
- $I \not\models_O B$ for any $B \in Neg$.

We have that

$$(i) I \not\models_O B$$

$$\text{iff } I \not\models \tau'(B) \text{ (by Lemma 7)}$$

$$\text{iff } E \not\models \tau'(B).$$

$$(ii) \gamma_{\mathcal{K}^s, I}^n \models_O A$$

$$\text{iff } E_n \cap HB_P \models_O A \text{ (by the inductive assumption)}$$

$$\text{iff } E_n \cap HB_P \models \tau'(A) \text{ (by Lemma 7)}$$

$$\text{iff } E_n \models \tau'(A).$$

It follows that $h \in \gamma_{\mathcal{K}^s, I}^{n+1}$ if and only if $h \in E_{n+1}$. Consequently $\gamma_{\mathcal{K}^s, I}^i = E_i \cap HB_P$ for any $i \geq 0$. ■

In the next theorem and corollary, we present the main results of this section, which extend Theorem 6 and Corollary 7 respectively.

Theorem 9 Let $\mathcal{K} = (O, P)$ be a dl-program such that $DL_P^? = \emptyset$ and $I \subseteq HB_P$. Then I is a strong answer set of \mathcal{K} if and only if $E = Th(I)$ is an extension of $\tau'(\mathcal{K})$.

Proof: (\Rightarrow) It is sufficient to show $E = \bigcup_{i \geq 0} E_i$ where E_i is defined as (8) for $\tau'(\mathcal{K})$ and E .

$$E = Th(I)$$

$$\Rightarrow E \equiv I$$

$$\Rightarrow E \equiv \gamma_{\mathcal{K}^s, I}^\infty \text{ (since } I \text{ is a strong answer set of } \mathcal{K})$$

$$\Rightarrow E \equiv \bigcup_{i \geq 0} E_i \cap HB_P \text{ (by Lemma 8)}$$

$$\Rightarrow E \equiv \bigcup_{i \geq 0} E_i \text{ (since } E_i \subseteq Th(HB_P))$$

$\Rightarrow E = \bigcup_{i \geq 0} E_i$
 $\Rightarrow E$ is an extension of $\tau'(\mathcal{K})$.
 $(\Leftarrow) E$ is an extension of $\tau'(\mathcal{K})$
 $\Rightarrow E = \bigcup_{i \geq 0} E_i$ where E_i is defined as (8) for $\tau'(\mathcal{K})$ and E
 $\Rightarrow Th(I) = \bigcup_{i \geq 0} E_i$
 $\Rightarrow Th(I) \cap HB_P = \left(\bigcup_{i \geq 0} E_i \right) \cap HB_P$
 $\Rightarrow I = \bigcup_{i \geq 0} (E_i \cap HB_P)$
 $\Rightarrow I = \gamma_{\mathcal{K}^s, I}^\infty$ (by Lemma 8)
 $\Rightarrow I = lfp(\gamma_{\mathcal{K}^s, I})$
 $\Rightarrow I$ is a strong answer set of \mathcal{K} . ■

Corollary 10 *Let $\mathcal{K} = (O, P)$ be a dl-program and $I \subseteq HB_P$. Then I is a strong answer set of \mathcal{K} if and only if $Th(\pi(I))$ is an extension of $\tau'(\pi(\mathcal{K}))$.*

Proof: I is a strong answer set of \mathcal{K}
 iff $\pi(I)$ is a strong answer set of $\pi(\mathcal{K})$ (by Theorem 3)
 iff $Th(\pi(I))$ is an extension of $\tau'(\pi(\mathcal{K}))$ (by Theorem 9). ■

Note that, for the dl-program \mathcal{K} in Example 9, we have $\tau'(\mathcal{K}) = (D, \emptyset)$ where D consists of

$$\frac{(p(a) \supset S(a)) \supset S(a) :}{p(a)}, \quad \frac{: \neg p(a)}{\neg p(a)}.$$

It is easy to see that $Th(\{\neg p(a)\})$ is the unique extension of $\tau'(\mathcal{K})$. As \mathcal{K} has two weak answer sets \emptyset and $\{p(a)\}$, the translation τ' alone does not preserve weak answer sets of dl-programs. However, one can further check that $\tau'(\sigma(\mathcal{K}))$ has exact two extensions $Th(\{p(a)\})$ and $Th(\{\neg p(a), \sigma_A\})$.

We show below that, combining with the translation σ , the translation τ' actually preserves weak answer sets.

Proposition 8 *Let $\mathcal{K} = (O, P)$ be a dl-program such that O is consistent, $DL_P^2 = \emptyset$ and all dl-atoms occurs negative in P , i.e., there for any rule $(h \leftarrow Pos, not Neg)$ of P , there is no dl-atom in Pos . Then an interpretation $I \subseteq HB_P$ is a weak answer set of \mathcal{K} iff $E = Th(I)$ is an extension of $\tau'(\mathcal{K})$.*

Proof: The proof is similar to the one of Proposition 7. ■

Together with Theorem 4, the propositions 6 and 8 imply a translation from dl-programs with consistent ontologies under the weak answer set semantics to default theories.

Corollary 11 *Let $\mathcal{K} = (O, P)$ be a dl-program where O is consistent. The following conditions are equivalent:*

- (i) *The interpretation $I \subseteq HB_P$ is a weak answer set \mathcal{K} .*
- (ii) *$Th(\pi(I'))$ is an extension of $\tau'(\pi(\sigma(\mathcal{K})))$ where $I' = I \cup \{\sigma_B \mid B \in DL_P \text{ and } I \not\models_O B\}$.*

Since there are no dl-atoms that occur positively in $\sigma(\mathcal{K})$, the translation $\sigma \cdot \pi$, i.e., applying σ first and then π , is polynomial. Consequently the combination $\sigma \cdot \pi \cdot \pi'$ is polynomial as well. Therefore, we have a polynomial, faithful and modular translation from dl-programs under the weak answer set semantics to default theories.

4.4 Under the well-supported semantics

To avoid circular justifications in some weak and strong answer sets of dl-programs, recently well-supported semantics for dl-programs was proposed (Shen, 2011). In what follows, we will show that, under the weakly well-supported answer set semantics, dl-programs can be translated into default theories by an extension of the translation τ above. In particular, the translation is polynomial, faithful and modular. Let us recall the basic notions and notations of well-supported semantics below.

Let $\mathcal{K} = (O, P)$ be a dl-program, E and I two sets of atom in HB_P with $E \subseteq I$. The notion that E up to I satisfies an atom (or a dl-atom, or their negation by default) l under O , written $(E, I) \models_O l$, is as follows:

- $(E, I) \models_O p$ if $p \in E$; $(E, I) \models_O \text{not } p$ if $p \notin I$, where p is an atom;
- $(E, I) \models_O A$ if for every F with $E \subseteq F \subseteq I$, $F \models_O A$; $(E, I) \models_O \text{not } A$ if there is no F with $E \subseteq F \subseteq I$ such that $F \models_O A$, where A is a dl-atom.

The notion “up to satisfaction” is extended for a set of atoms dl-atoms, and their negation by default in a standard manner¹¹. The operator $\mathcal{T}_{\mathcal{K}} : (2^{HB_P} \times 2^{HB_P}) \rightarrow 2^{HB_P}$ is defined as:

$$\mathcal{T}_{\mathcal{K}}(E, I) = \{a \mid (a \leftarrow \text{Body}) \in P \text{ and } (E, I) \models_O \text{Body}\}$$

where $E \subseteq I$. It has been shown that if I is a model of \mathcal{K} , then the operator is monotone in the sense that for every, $E_1 \subseteq E_2 \subseteq I$, $\mathcal{T}_{\mathcal{K}}(E_1, I) \subseteq \mathcal{T}_{\mathcal{K}}(E_2, I)$. As the operator is also continuous in this sense (thanks to compactness of answering DL queries), for any model I of \mathcal{K} the monotone sequence $\langle \mathcal{T}_{\mathcal{K}}^i(\emptyset, I) \rangle_i^\infty$, where $\mathcal{T}_{\mathcal{K}}^0(\emptyset, I) = \emptyset$, $\mathcal{T}_{\mathcal{K}}^{i+1}(\emptyset, I) = \mathcal{T}_{\mathcal{K}}(\mathcal{T}_{\mathcal{K}}^i(\emptyset, I), I)$, $i \geq 0$, converges to a fixpoint denoted $\mathcal{T}_{\mathcal{K}}^\infty(\emptyset, I)$.

In the rest of this paper, for convenience we will use the term *level mapping justification* to refer to the existence of such a fixpoint, borrowing a concept from a similar characterization for normal logic programs (Fages, 1994) as well as for weight constraint programs (Liu & You, 2010).

A model I of \mathcal{K} is a *weakly (resp. strongly) well-supported answer set* of \mathcal{K} if I coincides with the fixpoint $\mathcal{T}_{\mathcal{K}^I}^\alpha(\emptyset, I)$ (resp. $\mathcal{T}_{\mathcal{K}^I}^\infty(\emptyset, I)$, where $\mathcal{K}^I = (O, P^I)$ and

$$P^I = \{a \leftarrow \text{Pos} \mid (a \leftarrow \text{Pos}, \text{not Neg}) \in P \text{ and } I \not\models_O B \text{ for every } B \in \text{Neg}\}.$$

As the next proposition shows, the strongly well-supported answer set semantics coincides with the strong answer set semantics for the dl-programs that mention no nonmonotonic dl-atoms.

Proposition 9 *Let $\mathcal{K} = (O, P)$ be a dl-program with $DL_P^? = \emptyset$ and $I \subseteq HB_P$ a model of \mathcal{K} . Then I is a strong answer set of \mathcal{K} iff I is a strongly well-supported answer set of \mathcal{K} .*

Proof: (\Leftarrow) This direction is obvious since, for any dl-program, each strongly well-supported answer set is a weakly well-supported answer sets (Corollary 3 of (Shen, 2011)) and each weakly well-supported answer set is a strong answer set (Theorem 6 in (Shen, 2011)).

(\Rightarrow) It suffices to show $I \subseteq \mathcal{T}_{\mathcal{K}}^\alpha(\emptyset, I)$. Since $I = \gamma_{\mathcal{K}^s, I}^\infty$. We only need to show inductively, $\gamma_{\mathcal{K}^s, I}^n \subseteq \mathcal{T}_{\mathcal{K}}^n(\emptyset, I)$ for any $n \geq 0$.

11. The notion of “up to satisfaction” is very similar to that of “conditional satisfaction” in logic programs with abstract constraints (Son, Pontelli, & Tu, 2007).

Base: it is evident for $n = 0$.

Step: Let us consider the case $n + 1$. For any atom $h \in \gamma_{\mathcal{K}^s, I}^{n+1}$, there must exist a rule $(h \leftarrow Pos, not Neg)$ in P s.t.

- $\gamma_{\mathcal{K}^s, I}^n \models_O A$ for any $A \in Pos$ since $DL_P^? = \emptyset$, and
- $I \not\models_O B$ for any $B \in Neg$

Note that all dl-atoms in P are monotonic. It follows that $(\gamma_{\mathcal{K}^s, I}^n, I) \models_O A$ for any $A \in Pos$ and thus $(\mathcal{T}_{\mathcal{K}}^n(\emptyset, I), I) \models_O A$ by the inductive assumption. On the other hand, since $I \not\models_O B$ and B is monotonic, we have that, $I' \not\models_O B$ for any $I' \subseteq I$. It implies $(\emptyset, I) \models_O not B$ and thus $(\mathcal{T}_{\mathcal{K}}^n(\emptyset, I), I) \models_O not B$. Consequently $h \in \mathcal{T}_{\mathcal{K}}^{n+1}(\emptyset, I)$ and then $I \subseteq \mathcal{T}_{\mathcal{K}}^{\alpha}(\emptyset, I)$. It follows that I is a strongly well-supported answer set of \mathcal{K} . ■

Before presenting a translation under weakly well-supported answer set semantics, let us reconsider the dl-program \mathcal{K} in Example 8. Recall that the dl-program \mathcal{K} has a strong answer set $\{p(a), q(a)\}$ and the unique extension of $\tau(\mathcal{K})$ is $Th(\tau(\emptyset))$. Actually, \emptyset is not a model of \mathcal{K} at all. We can check that \mathcal{K} has neither a weakly well-supported answer set, nor a strongly well-supported answer set. Thus the translation τ works neither for weakly nor for strongly well-supported answer set semantics of dl-programs.

Surprisingly, a small addition to our default logic encoding will result in a one-one correspondence between the weakly well-supported answer sets of a dl-program and the corresponding default extensions, for arbitrary dl-programs. Below, we consider the dl-programs whose ontology component is consistent. Formally, given a dl-program $\mathcal{K} = (O, P)$ where O is consistent, we define $\tau^*(\mathcal{K}) = (D, W)$ where τ^* is exactly the same as τ except that D includes, for each $p(\vec{c}) \in HB_P$, the default

$$\frac{: \neg p(\vec{c})}{\neg p(\vec{c})}.$$

It is evident that any extension E of $\tau^*(\mathcal{K})$ is equivalent to $\tau(O) \cup I \cup \{\neg \alpha \mid \alpha \in HB_P \setminus I\}$ for some $I \subseteq HB_P$.

Example 11 Let us reconsider the dl-program \mathcal{K} in Example 8. The default theory $\tau^*(\mathcal{K}) = (D, \tau(\emptyset))$ where D consists of the ones produced by τ and additionally the ones

$$\frac{: \neg p(a)}{\neg p(a)}, \quad \frac{: \neg q(a)}{\neg q(a)}.$$

It is not difficult to check that $\tau^*(\mathcal{K})$ has no extension. This example also demonstrates that τ^* does not preserve the strong answer sets of dl-programs as \mathcal{K} has a strong answer set $\{p(a), q(a)\}$.

In the following, given a dl-program $\mathcal{K} = (O, P)$ and $I \subseteq HB_P$, we denote $\bar{I} = HB_P \setminus I$ and $\neg I = \{\neg \alpha \mid \alpha \in I\}$ for convenience.

Lemma 9 Let M_1 and M_2 be two sets of atoms such that $M_1 \cap M_2 = \emptyset$, ψ_i, φ_i ($1 \leq i \leq n$) and ϕ are formulas not mentioning the predicate p_1, p_2 and the predicates occurring in $M_1 \cup M_2$. Then

$$\bigwedge M_1 \wedge \bigwedge \neg M_2 \wedge \bigwedge_{1 \leq i \leq n} ((p_1(\vec{c}_i) \supset \psi_i) \wedge (\neg p_2(\vec{c}_i) \supset \varphi_i)) \models \phi \text{ iff } \bigwedge_{p_1(\vec{c}_i) \in M_1} \psi_i \wedge \bigwedge_{p_2(\vec{c}_j) \in M_2} \varphi_j \models \phi.$$

Proof: The direction from right to left is obvious. Let us consider the other direction. Suppose there is an interpretation \mathcal{I} such that $\mathcal{I} \models \bigwedge_{p_1(\vec{c}_i) \in M_1} \psi_i \wedge \bigwedge_{p_2(\vec{c}_j) \in M_2} \varphi_j$ but $\mathcal{I} \not\models \phi$, by which we have $\mathcal{I} \not\models \bigwedge M_1 \wedge \bigwedge \neg M_2 \wedge \bigwedge_{1 \leq i \leq n} ((p_1(\vec{c}_i) \supset \psi_i) \wedge (\neg p_2(\vec{c}_i) \supset \varphi_i))$. It follows that $\mathcal{I} \not\models \bigwedge M_1 \wedge \bigwedge \neg M_2 \wedge \bigwedge_{p_1(\vec{c}_i) \notin M_1} (p_1(\vec{c}_i) \supset \psi_i) \wedge \bigwedge_{p_2(\vec{c}_j) \notin M_2} (\neg p_2(\vec{c}_j) \supset \varphi_j)$. We construct the interpretation \mathcal{I}' that is same to \mathcal{I} except that

- $\mathcal{I}' \models \bigwedge M_1$, and $\mathcal{I}' \models \bigwedge \neg M_2$,
- $\mathcal{I}' \not\models p_1(\vec{c}_i)$ for every $p_1(\vec{c}_i) \notin M_1$, and
- $\mathcal{I}' \models p_2(\vec{c}_j)$ for every $p_2(\vec{c}_j) \notin M_2$.

It is clear that $\mathcal{I}' \models \bigwedge_{p_1(\vec{c}_i) \in M_1} \psi_i \wedge \bigwedge_{p_2(\vec{c}_j) \in M_2} \varphi_j$ and $\mathcal{I}' \not\models \phi$. However, we have $\mathcal{I}' \models \phi$ by $\mathcal{I}' \models \bigwedge M_1 \wedge \bigwedge \neg M_2 \wedge \bigwedge_{1 \leq i \leq n} ((p_1(\vec{c}_i) \supset \psi_i) \wedge (\neg p_2(\vec{c}_i) \supset \varphi_i))$, a contradiction. \blacksquare

Lemma 10 Let $\mathcal{K} = (O, P)$ be a dl-program, $A = DL[\lambda; Q](\vec{t})$ a dl-atom and $I \subseteq HB_P$.

- (i) $I \models_O A$ iff $\tau(O) \cup I \cup \neg \bar{I} \models \tau(A)$.
- (ii) if $I' \subseteq I$ then $(I', I) \models_O A$ iff $\tau(O) \cup I' \cup \neg \bar{I} \models \tau(A)$.

Proof: For clarity and without loss of generality, let $\lambda = (S_1 \oplus p_1, S_2 \ominus p_2)$.

(i) We have that at first $\tau(O) \cup I \cup \neg \bar{I} \models \tau(A)$
 iff $\tau(O) \cup I \cup \neg \bar{I} \models (\bigwedge_{\vec{e} \in \vec{C}} (p_1(\vec{e}) \supset S_1(\vec{e})) \wedge (\bigwedge_{\vec{e} \in \vec{C}} (\neg p_2(\vec{e}) \supset \neg S_2(\vec{e}))) \supset Q(\vec{t})$
 iff $I \cup \neg \bar{I} \cup \{ \bigwedge_{\vec{e} \in \vec{C}} (p_1(\vec{e}) \supset S_1(\vec{e})) \} \cup \{ \bigwedge_{\vec{e} \in \vec{C}} (\neg p_2(\vec{e}) \supset \neg S_2(\vec{e})) \} \models \tau(O) \supset Q(\vec{t})$
 iff $\{ S_1(\vec{e}) \mid p_1(\vec{e}) \in I \} \cup \{ \neg S_2(\vec{e}) \mid p_2(\vec{e}) \notin I \} \models \tau(O) \supset Q(\vec{t})$ (By Lemma 9)
 iff $\tau(O) \cup \{ S_1(\vec{e}) \mid p_1(\vec{e}) \in I \} \cup \{ \neg S_2(\vec{e}) \mid p_2(\vec{e}) \notin I \} \models Q(\vec{t})$
 iff $O \cup \{ S_1(\vec{e}) \mid p_1(\vec{e}) \in I \} \cup \{ \neg S_2(\vec{e}) \mid p_2(\vec{e}) \notin I \} \models Q(\vec{t})$ (by Theorem 5, where \approx is taken as equality)
 iff $I \models_O A$.

(ii) (\Leftarrow) By $\tau(O) \cup I' \cup \neg \bar{I} \models \tau(A)$, we have that, for any F with $I' \subseteq F \subseteq I$, $\tau(O) \cup F \cup \neg \bar{I} \models \tau(A)$ which implies $\tau(O) \cup F \cup \neg \bar{F} \models \tau(A)$. Thus $F \models_O A$ by (i). Consequently $(I', I) \models_O A$.
 (\Rightarrow) Let $S = I \setminus I' = \{\alpha_1, \dots, \alpha_k\}$ and $J = \{1, \dots, k\}$. It is clear that $\neg S = \neg \bar{I}' \setminus \neg \bar{I}$. Note that for any F with $I' \subseteq F \subseteq I$, $F \models_O A$, which implies $\tau(O) \cup F \cup \neg \bar{F} \models \tau(A)$ by (i), i.e., for any $J' \subseteq J$, we have that

$$I' \cup \{ \alpha_i \mid i \in J' \} \cup \{ \neg \alpha_j \mid j \in J \setminus J' \} \cup \neg \bar{I} \models O \supset \tau(A)$$

which implies that

$$\bigvee_{J' \subseteq J} \left(\bigwedge_{i \in J'} \alpha_i \wedge \bigwedge_{j \in J \setminus J'} \neg \alpha_j \right) \models I' \wedge \neg \bar{I} \supset (\tau(O) \supset \tau(A)).$$

Thus we have, by Lemma 3

$$\bigwedge_{i \in J} (\alpha_i \vee \neg \alpha_i) \models I' \wedge \neg \bar{I} \supset (\tau(O) \supset \tau(A))$$

i.e.,

$$I' \cup \neg \bar{I} \models \tau(O) \supset \tau(A).$$

Consequently we have $\tau(O) \cup I' \cup \neg \bar{I} \models \tau(A)$. ■

It is easy to see that if A is an atom and O is consistent, then both (i) and (ii) of the above lemma hold.

Lemma 11 *Let $\mathcal{K} = (O, P)$ be a dl-program where O is consistent and $I \subseteq HB_P$ is a model of \mathcal{K} . Then we have that, for any $i \geq 0$, E_i is consistent where E_i is defined as (8) for $\tau^*(\mathcal{K})$ and $E = Th(\tau(O) \cup I \cup \neg \bar{I})$.*

Proof: It is sufficient to show that $E_i \cap HB_P \subseteq I$ for every $i \geq 0$.

Base: It is clear for $i = 0$ since O is consistent. For the case $i = 1$, we have that $\neg \bar{I} \subseteq E_1$. If E_1 is inconsistent then there must exist a rule $(h \leftarrow Pos, not Neg)$ in P such that

- $h \in \bar{I}$,
- $E_0 \models \tau(A)$ for every $A \in Pos$, and
- $E \not\models \tau(B)$ for every $B \in Neg$.

It is evident that $I \not\models_O B$ for every $B \in Neg$ by (i) of Lemma 10. And note that

$$\begin{aligned} E_0 &\models \tau(A) \\ \Rightarrow \tau(O) &\models \tau(A) \\ \Rightarrow \tau(O) \cup I \cup \neg \bar{I} &\models \tau(A) \\ \Rightarrow I &\models_O A \text{ by (i) of Lemma 10.} \end{aligned}$$

It follows that $h \in I$ since I is a model of \mathcal{K} . It contradicts with $h \in \bar{I}$.

Step: Suppose E_n is consistent where $n \geq 1$. For any atom $h \in HB_P$, $h \in E_{n+1}$ if and only if there exists a rule $(h' \leftarrow Pos', not Neg')$ in P such that

- $E_n \models \tau(A')$ for any $A' \in Pos'$, and
- $E \not\models \tau(B')$ for any $B' \in Neg'$.

It is clear that $I \not\models_O B'$ for any $B' \in Neg'$ by (i) of Lemma 10. Since E_n is consistent by the inductive assumption, we have that $(E_n \cap HB_P) \cap \bar{I} = \emptyset$ by $\neg \bar{I} \subseteq E_n$. Thus it follows that

$$\begin{aligned} E_n &\models \tau(A') \\ \Rightarrow \tau(O) \cup (E_n \cap HB_P) \cup \neg \bar{I} &\models \tau(A') \\ \Rightarrow \tau(O) \cup I \cup \neg \bar{I} &\models \tau(A') \text{ since } E_n \cap HB_P \subseteq I \\ \Rightarrow I &\models_O A' \text{ by (i) of Lemma 10.} \end{aligned}$$

It implies that $h \in I$ since I is a model of \mathcal{K} . Thus E_{n+1} is consistent. ■

Lemma 12 *Let $\mathcal{K} = (O, P)$ be a dl-program where O is consistent and $I \subseteq HB_P$ a model of \mathcal{K} . Then we have that, for any $i \geq 0$,*

- (i) $\mathcal{T}_{\mathcal{K}^I}^i(\emptyset, I) \subseteq E_{i+1} \cap HB_P$, and
- (ii) $E_i \cap HB_P \subseteq \mathcal{T}_{\mathcal{K}^I}^i(\emptyset, I)$

where E_i is defined as (8) for $\tau^*(\mathcal{K})$ and $E = Th(\tau(O) \cup I \cup \neg\bar{I})$.

Proof: We prove (i) and (ii) by induction on i .

(i) Base: It is evident for $i = 0$.

Step: Suppose it holds for $i = n$ where $n \geq 0$. For any atom $h \in HB_P$, we have that $h \in \mathcal{T}_{\mathcal{K}^I}^{n+1}(\emptyset, I)$ if and only if there exists a rule $(h \leftarrow Pos, not Neg)$ in P such that

- $(\mathcal{T}_{\mathcal{K}^I}^n(\emptyset, I), I) \models_O A$ for any $A \in Pos$, and
- $I \not\models_O B$ for any $B \in Neg$.

By (i) of Lemma 10, $I \not\models_O B$ iff $E \not\models \tau^*(B)$, and by (ii) of Lemma 10, we have

$$\begin{aligned} & (\mathcal{T}_{\mathcal{K}^I}^n(\emptyset, I), I) \models_O A \\ \Rightarrow & \tau(O) \cup \mathcal{T}_{\mathcal{K}^I}^n(\emptyset, I) \cup \neg\bar{I} \models \tau(A) \\ \Rightarrow & \tau(O) \cup (E_{n+1} \cap HB_P) \cup \bar{I} \models \tau(A) \text{ (by the induction assumption)} \\ \Rightarrow & E_{n+1} \models \tau(A) \text{ (since } \tau(O) \cup \neg\bar{I} \subseteq E_{n+1}) \\ \Rightarrow & h \in E_{n+2}. \end{aligned}$$

(ii) Base: It is clear for $i = 0$. Let us consider the case $i = 1$. For any atom $h \in E_1 \cap HB_P$, there exists a rule $(h \leftarrow Pos, not Neg)$ in P such that

- $E_0 \models \tau(A)$ for any $A \in Pos$, and
- $E \not\models \tau(B)$ for any $B \in Neg$.

By $E_0 \models \tau(A)$, we have $O \models \tau(A)$. Thus $\tau(O) \cup I' \cup \neg\bar{I} \models \tau(A)$ for any I' such that $I' \subseteq I$. It implies $(\emptyset, I) \models_O A$ by (ii) of Lemma 10. By (i) of Lemma 10 and $E \not\models \tau(B)$, it is evident $I \not\models_O B$. It follows that $h \in \mathcal{T}_{\mathcal{K}^I}^1(\emptyset, I)$.

Step: Suppose it holds for $i = n$ where $n \geq 1$. For any atom $h' \in (E_{n+1} \cap HB_P)$, there exists a rule $(h' \leftarrow Pos', not Neg')$ in P such that

- $E_n \models \tau(A')$ for any $A' \in Pos'$, and
- $E \not\models \tau(B')$ for any $B' \in Neg'$.

Since I is a model of \mathcal{K} , E_n is consistent by Lemma 11. Note that for any $n \geq 1$ and $\tau(O) \cup \neg\bar{I} \subseteq E_n$. It implies $E_n \cap HB_P \subseteq I$. We have that

$$\begin{aligned} & E_n \models \tau(A') \\ \Rightarrow & O \cup (E_n \cap HB_P) \cup \neg\bar{I} \models \tau(A') \\ \Rightarrow & (E_n \cap HB_P, I) \models_O A' \text{ by (ii) of Lemma 10} \\ \Rightarrow & (\mathcal{T}_{\mathcal{K}^I}^n(\emptyset, I), I) \models_O A' \text{ by the inductive assumption and the monotonicity of } \mathcal{T}_{\mathcal{K}^I}. \end{aligned}$$

Notice again that $E \not\models \tau(B')$ implies $I \not\models_O B'$ by (i) of Lemma 10. Thus it follows that $h' \in \mathcal{T}_{\mathcal{K}^I}^{n+1}(\emptyset, I)$.

This completes the proof. ■

Please note that it does not generally hold that $\mathcal{T}_{\mathcal{K}^I}^i(\emptyset, I) = E_i \cap HB_P$ in the above lemma. For instance, let us consider the dl-program $\mathcal{K} = (\emptyset, P)$ where P consists of

$$p(a) \leftarrow DL[S \oplus p, S' \ominus q; S \sqcup \neg S'](a).$$

Let $I = \{p(a)\}$. It is obvious that $p(a) \in \mathcal{T}_{\mathcal{K}^I}(\emptyset, I)$, i.e. $p(a) \in \mathcal{T}_{\mathcal{K}^I}^1(\emptyset, I)$. However, it is clear that $E_0 \not\models \tau(A)$ since $E_0 = Th(\tau(\emptyset))$ where $A = DL[S \oplus p, S' \ominus q; S \sqcup \neg S'](a)$. Thus $p(a) \notin E_1$.

The theorem below shows that the polynomial and modular translation τ^* preserves the weakly well-supported answer set semantics of dl-programs. Thus it is faithful.

Theorem 12 *Let $\mathcal{K} = (O, P)$ be a dl-program where O is consistent and $I \subseteq HB_P$ a model of \mathcal{K} . Then we have that I is a weakly well-supported answer set of \mathcal{K} iff $E = Th(\tau(O) \cup I \cup \neg \bar{I})$ is an extension of $\tau^*(\mathcal{K})$.*

Proof: (\Rightarrow) To show $E = \bigcup_{i \geq 0} E_i$ where E_i is defined as (8) for E and $\tau^*(\mathcal{K})$, it is sufficient to show $E \cap HB_P = (\bigcup_{i \geq 0} E_i) \cap HB_P$ since $\tau(O) = E_0$, $\neg \bar{I} \subseteq E_1$ and E_i is consistent for any $i \geq 0$ by Lemma 11.

For any $h \in HB_P$, it is clear that $h \in E \cap HB_P$ iff $h \in I$ iff $h \in \mathcal{T}_{\mathcal{K}^I}^n(\emptyset, I)$ for some $n \geq 0$ since $I = \mathcal{T}_{\mathcal{K}^I}^\alpha$.

On the one hand, $h \in \mathcal{T}_{\mathcal{K}^I}^n(\emptyset, I)$ implies $h \in E_{n+1} \cap HB_P$ by (i) of Lemma 12 and then $h \in \bigcup_{i \geq 0} (E_i \cap HB_P)$, i.e. $h \in (\bigcup_{i \geq 0} E_i) \cap HB_P$. On the other hand $h \in (\bigcup_{i \geq 0} E_i) \cap HB_P$ implies $h \in \bigcup_{i \geq 0} (E_i \cap HB_P)$, i.e. $h \in E_n \cap HB_P$ for some $n \geq 0$. It follows that $h \in \mathcal{T}_{\mathcal{K}^I}^n(\emptyset, I)$ by (ii) of Lemma 12. Thus $h \in I$ and the $h \in E \cap HB_P$.

Consequently, we have $E \cap HB_P = (\bigcup_{i \geq 0} E_i) \cap HB_P$.

(\Leftarrow) By Theorem 3 of (Shen, 2011), it is clear that $\mathcal{T}_{\mathcal{K}^I}^\alpha(\emptyset, I) \subseteq I$. We only need to show $I \subseteq \mathcal{T}_{\mathcal{K}^I}^\alpha(\emptyset, I)$. For any atom $h \in I$, we have that $h \in E$

$\Rightarrow h \in (\bigcup_{i \geq 0} E_i) \cap HB_P$ since $E = \bigcup_{i \geq 0} E_i$
 $\Rightarrow h \in E_n \cap HB_P$ for some $n \geq 0$ since E_i is consistent for any $i \geq 0$
 $\Rightarrow h \in \mathcal{T}_{\mathcal{K}^I}^n(\emptyset, I)$ by (ii) of Lemma 12
 $\Rightarrow h \in \mathcal{T}_{\mathcal{K}^I}^\alpha(\emptyset, I)$.

This completes the proof. ■

Together with Theorem 3 and Proposition 9, the above theorem implies another translation from dl-programs to default theories that preserves the strong answer set semantics.

Corollary 13 *Let $\mathcal{K} = (O, P)$ be a dl-program where O is consistent and $I \subseteq HB_P$.*

- *If $DL_P^? = \emptyset$ then I is a strong answer set of \mathcal{K} iff $Th(\tau(O) \cup I \cup \neg \bar{I})$ is an extension of $\tau^*(\mathcal{K})$ iff I is a strongly well-supported answer set of \mathcal{K} .*
- *I is a strong answer set of \mathcal{K} iff $Th(\tau(O) \cup \pi(I) \cup \neg \pi(\bar{I}))$ is an extension of $\tau^*(\pi(\mathcal{K}))$.*

We note that the translation τ^* does not preserve the strongly well-supported answer sets of dl-programs. For instance, let us consider the dl-program \mathcal{K}_1 in Example 4. It is easy to see that the only strongly well-supported answer set of \mathcal{K}_1 is \emptyset , while $\tau^*(\mathcal{K}_1)$ has two extensions $Th(\{\neg p(a)\} \cup \tau(\emptyset))$ and $Th(\{p(a)\} \cup \tau(\emptyset))$. However, the translation τ^* does preserve the strongly well-supported answer sets for a highly relevant class of dl-programs as illustrated by the next proposition. The following lemma is a generalization of Corollary 4 of (Shen, 2011).

Lemma 13 *Let $\mathcal{K} = (O, P)$ be a dl-program such that, for every rule of the form (3) in P , the dl-atom B is monotonic if $B \in \text{Neg}$, and $I \subseteq HB_P$. Then I is a weakly well-supported answer set of \mathcal{K} iff I is a strongly well-supported answer set of \mathcal{K} .*

Proof: The direction from right to left is implied by Corollary 2 of (Shen, 2011) which asserts this for arbitrary dl-programs. To show the other direction, it suffices to prove

$$\mathcal{T}_{\mathcal{K}^I}^n(\emptyset, I) = \mathcal{T}_{\mathcal{K}}^n(\emptyset, I)$$

for every $n \geq 0$ by induction.

Base: the case $n = 0$ is obvious.

Step: suppose the statement holds for n and consider the case $n + 1$. For any atom $h \in HB_P$, we have that $h \in \mathcal{T}_{\mathcal{K}^I}^{n+1}(\emptyset, I)$ iff there exists a rule $r \in P$ such that

- $(\mathcal{T}_{\mathcal{K}^I}^n(\emptyset, I), I) \models_O A$ for any $A \in Pos(r)$, and
- $I \not\models_O B$ for any $B \in Neg(r)$.

Recall that I is a weakly well-supported answer set of \mathcal{K} , by which $(\mathcal{T}_{\mathcal{K}^I}^n(\emptyset, I), I) \subseteq I$. It shows that (a) if B is an atom then $I \not\models_O B$ iff $(\mathcal{T}_{\mathcal{K}^I}^n(\emptyset, I), I) \not\models_O B$, and (b) if B is a monotonic dl-atom then $I \not\models_O B$ iff $(\mathcal{T}_{\mathcal{K}^I}^n(\emptyset, I), I) \not\models_O B$ as well. It follows that $h \in \mathcal{T}_{\mathcal{K}^I}^{n+1}(\emptyset, I)$ iff $h \in \mathcal{T}_{\mathcal{K}}^{n+1}(\emptyset, I)$ by inductive assumption. \blacksquare

Proposition 10 *Let $\mathcal{K} = (O, P)$ be a dl-program such that, for every rule of the form (3) in P , the dl-atom B is monotonic if $B \in Neg$, and $I \subseteq HB_P$. Then I is a strongly well-supported answer set of \mathcal{K} iff $E = Th(\tau(O) \cup I \cup \neg \bar{I})$ is an extension of $\tau^*(\mathcal{K})$.*

Proof: In terms of the definition of weakly and strong well-supported answer sets, it is obvious that I is a strongly well-supported answer set of \mathcal{K}

iff I is a weakly well-supported answer set of \mathcal{K} by Lemma 13

iff $Th(\tau(O) \cup I \cup \neg \bar{I})$ is an extension of $\tau^*(\mathcal{K})$ by Theorem 12. \blacksquare

At a first glance, in order to preserve the strongly well-supported answer set semantics, one might suggest to “shift” $\neg\tau(\cdot)$ for all dl-atoms from justification to the premise of a default. This does not work, as illustrated by the dl-program $\mathcal{K} = (\emptyset, P)$ where $P = \{p(a) \leftarrow not DL[S \oplus p, S'](a)\}$. It is obvious that \mathcal{K} has a strongly well-supported answer set $\{p(a)\}$. But according to the suggestion, we would have the default theory $\Delta = (D, W)$ where $W = \tau(\emptyset)$ and D consists of

$$\frac{\neg((p(a) \supset S(a)) \supset S'(a))}{p(a)}, \quad \frac{\neg p(a)}{\neg p(a)}.$$

Its unique extension is $Th(\{\neg p(a)\} \cup \tau(\emptyset))$, which does not correspond to any strongly well-supported answer set of \mathcal{K} . The reader can further check the dl-program \mathcal{K}_1 in Example 1 and see that “shifting” $\tau(\cdot)$ for all dl-atoms from premise to justification of a default does not work under the weak answer set semantics either.

For general ontologies (consistent or inconsistent), we can slightly modify the translation π^* similarly as τ to τ' , to obtain a transformation $\pi^{*'}$ and derive analogous results for it.

Let us now summarize the translations in Table 1. Note that all the translations τ, τ^*, σ and π are faithful and modular, and the first three are polynomial. In addition, π is polynomial relative to the knowledge of the non-monotonic dl-atoms $DL_P^?$, and thus e.g. polynomial for normal dl-programs. Table 1 shows that, for canonical dl-programs with consistent ontologies, we have polynomial,

Table 1: Translations from dl-programs with consistent ontologies to default theories

	WAS	SAS	WWAS	SWAS
Canonical dl-programs	$\sigma \cdot \tau$	τ / τ^*	τ^*	τ^*
Normal dl-programs	$\sigma \cdot \pi \cdot \tau$	$\pi \cdot (\tau / \tau^*)$	τ^*	–
Arbitrary dl-programs	$\sigma \cdot \pi \cdot \tau$	$\pi \cdot (\tau / \tau^*)$	τ^*	–

–: unknown; WAS: weak answer sets; SAS: strong answer sets;
 WWAS: weakly well-supported answer sets; SWAS: strongly well-supported answer sets.

faithful and modular translations for all the semantics, weak answer sets, strong answer sets, weakly well-supported answer sets and strongly well-supported answer sets.

In addition, under weak answer set and weakly well-supported answer set semantics, all the translations are polynomial, faithful and modular as well. One should note that, for normal dl-programs, the translation is also polynomial, faithful and modular. There are two unsolved problems, both involving the question whether there exist translations from dl-programs to default theories preserving strongly well-supported answer sets. In Table 1, it is assumed that dl-programs have consistent ontologies. To remove this assumption, it is sufficient to replace τ (resp., τ^*) with τ' (resp., τ'^*).

5. Related Work

Recently, there are some extensive interests in the FLP semantics for various kinds of logic programs (Faber, Pfeifer, & Leone, 2011; Bartholomew, Lee, & Meng, 2011; Truszczyński, 2010). Also, in formulating the well-founded semantics for dl-programs, Eiter *et al.* proposed a method to eliminate the constraint operator from dl-programs (Eiter et al., 2011). Moreover, there exist a number of formalisms integrating ontology and (nonmonotonic) rules for the semantics web that can somehow be used to embed dl-programs. In this section we will relate our work with these approaches.

5.1 FLP-answer sets of dl-programs

Dl-programs have been extended to HEX programs that combine answer set programs with higher-order atoms and external atoms (Eiter, Ianni, Schindlauer, & Tompits, 2005). In particular, external atoms can refer, as dl-atoms in dl-programs, to concepts belonging to a classical knowledge base or an ontology. In such a case one can compare the semantics of the HEX program with that of the corresponding dl-program. The semantics of HEX programs is based on the notion of FLP-reduct (Faber, Leone, & Pfeifer, 2004). We also note that the semantics of dl-programs has been investigated from the perspective of the quantified logic of here-and-there (Fink & Pearce, 2010). For comparison purpose, we rephrase the FLP-answer set semantics of dl-programs according to (Eiter et al., 2005) in our setting.

Let $\mathcal{K} = (O, P)$ be a dl-program and $I \subseteq HB_P$. The *FLP-reduct* of \mathcal{K} relative to I , written $\mathcal{K}^{f,I}$, is the dl-program (O, fP_O^I) where fP_O^I is the set of all rules of P whose bodies are satisfied by I relative to O . An interpretation I is an FLP-answer set of a dl-program \mathcal{K} if I is a minimal model of fP_O^I (relative to O). It has been shown that, for a dl-program $\mathcal{K} = (O, P)$, if P mentions no nonmonotonic dl-atoms, i.e., $DL_P^? = \emptyset$, then the FLP-answer sets of \mathcal{K} coincide with the strong

answer sets of \mathcal{K} (cf. Theorem 5 of (Eiter et al., 2005)). Moreover, following the approach on (Wang et al., 2010), it can be shown that the FLP-answer sets of a dl-program are exactly the minimal strong answer sets of the dl-program.

Note that, given a dl-program $\mathcal{K} = (O, P)$, there are no nonmonotonic dl-atoms in $\pi(\mathcal{K})$. Thus the strong answer sets of $\pi(\mathcal{K})$ are exactly the FLP-answer sets of $\pi(\mathcal{K})$. In general however, since FLP-answer sets are minimal strong answer sets and not vice versa, and π preserves strong answer sets, it is clear that π does not preserve the FLP-answer sets of dl-programs. This can be seen from Example 4. This fact reinforces our argument that there is no transformation to eliminate the constraint operator from nonmonotonic dl-atoms such that the transformation preserves both strong answer sets and FLP-answer sets of dl-programs. It is still open to us whether there is a translation to eliminate the constraint operator from nonmonotonic dl-atoms while preserving the FLP-answer sets of dl-programs.

As illustrated by Example 8, the translations τ and τ^* from dl-programs into default theories do not preserve FLP-answer sets. In addition, the translation τ may induce some extensions that correspond neither to strong answer sets nor to FLP-answer sets. Recall that, for dl-programs mentioning no nonmonotonic dl-atoms, the strong answer sets coincide with the FLP-answer sets. By Theorem 9, the following Corollary is obvious.

Corollary 14 *Let $\mathcal{K} = (O, P)$ be a dl-program such that $DL_P^? = \emptyset$ and $I \subseteq HB_P$. Then I is an FLP-answer set of \mathcal{K} if and only if $Th(I)$ is an extension of $\tau'(\mathcal{K})$.*

Since the constraint operator is the only that causes a dl-atom to be nonmonotonic, it follows that for dl-programs without the constraint operator, the strong answer set semantics and the FLP-answer set semantics can both be captured by default logic via a polynomial time transformation.

5.2 Eliminating the constraint operator for well-founded semantics

To the best of our knowledge, there is only one proposal to remove the constraint operator in dl-programs, for the definition of a well-founded semantics for dl-programs (Eiter et al., 2011). In fact, our translation draws ideas from theirs in order to preserve strong answer sets of dl-programs. However, there are subtle differences which make them significantly different in behaviors. Let us denote their transformation by π' . Given a dl-program $\mathcal{K} = (O, P)$ and a dl-rule $r \in P$, $\pi'(r)$ consists of

- (1) if $S \ominus p$ occurs in a dl-atom of r , then $\pi'(r)$ includes the instantiated rules obtained from

$$\bar{p}(\vec{X}) \leftarrow \text{not } DL[S' \oplus p; S'](\vec{X}).$$

where S' is a fresh concept (resp., role) name if S is a concept (resp., role) name, \vec{X} is a tuple of distinct variables matching the arity of p ,

- (2) $\pi'(r)$ includes the rule obtained from r by replacing each “ $S \ominus p$ ” with “ $\neg S \oplus \bar{p}$ ”¹². Let us denote by $\pi'(A)$ the result obtained from A by replacing every $S \ominus p$ with $\neg S \oplus \bar{p}$ where A is an atom or dl-atom.

12. It is “ $S \odot \bar{p}$ ” according to (Eiter et al., 2011) which is equivalent to “ $\neg S \oplus \bar{p}$ ”.

Similarly, $\pi'(\mathcal{K}) = (O, \pi'(P))$ where $\pi'(P) = \bigcup_{r \in P} \pi'(r)$. Let us consider the dl-program \mathcal{K}_2 in Example 1, $\pi'(P_2)$ consists of

$$\begin{aligned} p(a) &\leftarrow DL[S \oplus p, \neg S' \oplus \bar{q}; S \sqcap \neg S'](a), \\ \bar{q}(a) &\leftarrow not DL[S'' \oplus q; S''](a). \end{aligned}$$

It is not difficult to verify that $\pi'(\mathcal{K}_2)$ has a unique strong answer set $\{\bar{q}(a)\}$. Thus, π' loses a strong answer set, as $\{p(a)\}$ is a strong answer set of \mathcal{K}_2 but there is no corresponding strong answer set for $\pi'(\mathcal{K}_2)$.

The translation π' may even remove FLP-answer sets, as illustrated by the next example. Consider the dl-program \mathcal{K} in Example 8. It is not difficult to verify that the unique FLP-answer set of \mathcal{K} is $\{p(a), q(a)\}$. However we have $\pi'(\mathcal{K}) = (\emptyset, \pi'(P))$ where $\pi'(P)$ consists of

$$\begin{aligned} p(a) &\leftarrow q(a), \\ q(a) &\leftarrow DL[S_1 \oplus p, \neg S_2 \oplus \bar{q}; S_1 \sqcup \neg S_2](a), \\ \bar{q}(a) &\leftarrow not DL[S' \oplus q, S'](a). \end{aligned}$$

Interested readers can check that $\pi'(\mathcal{K})$ has no FLP-answer sets. Note that since any FLP-answer set is a strong answer set, this is another example where a strong answer set is removed by the translation.

The discussion above leads to a related question - whether the translation π' introduces extra strong answer sets, for a given dl-program $\mathcal{K} = (O, P)$. Note that in our translation π , for a predicate p we use predicate π_p to denote the opposite of p , while in the translation π' , the symbol \bar{p} is used. After reconciling this name difference, we see that the rule $\bar{p}(\vec{X}) \leftarrow not DL[S' \oplus p; S'](\vec{X})$ in the translation π' , where S' is a fresh concept or role name, is equivalent to rule (6) in the translation π . Then, the only difference is to apply “double negation” in the case of π to positive nonmonotonic dl-atoms. Given a dl-program \mathcal{K} , suppose an interpretation I is a strong answer set of $\pi'(\mathcal{K})$. Then I is the least model of $\pi'(\mathcal{K})^{s,I}$. It is not difficult to show that, in the fixpoint construction, for any atom $p \in HB_{\pi'(P)}$, p is derivable using $\pi'(\mathcal{K})^{s,I}$ if and only if p is derivable using $\pi(\mathcal{K})^{s,I}$. Therefore, I , possibly plus some atoms in the form of π_A , yields a strong answer set of $\pi(\mathcal{K})^{s,I}$.

Proposition 11 *Let $\mathcal{K} = (O, P)$ be a dl-program and $I \subseteq HB_{\pi'(P)}$ a strong answer set of $\pi'(\mathcal{K})$. Then $I \cap HB_P$ is a strong answer set of \mathcal{K} .*

Proof: Let $I^* = I \cap HB_P$, and we prove I^* is a strong answer set of \mathcal{K} . It is completed by showing $I^* = lfp(\gamma_{\mathcal{K}^{s,I^*}})$.

(\subseteq) We prove the direction by showing $HB_P \cap \gamma_{[\pi'(\mathcal{K})]^{s,I}}^k \subseteq lfp(\gamma_{\mathcal{K}^{s,I^*}})$ for any $k \geq 0$.

Base: It is trivial for $k = 0$.

Step: Suppose it holds for the case k . Let us consider the case $k + 1$. For any atom p in HB_P such that $p \in \gamma_{[\pi'(\mathcal{K})]^{s,I}}^{k+1}$, there exists a rule $(p \leftarrow Pos, not Neg)$ in P such that

- $\gamma_{[\pi'(\mathcal{K})]^{s,I}}^k \models_O \pi'(A)$ for any $A \in Pos$, and
- $I \not\models_O \pi'(B)$ for any $B \in Neg$.

It follows that

- If A is an atom or monotonic dl-atom then $HB_P \cap \gamma_{[\pi'(\mathcal{K})]^{s,I}}^k \models_O A$ by Lemma 1. It follows $lfp(\gamma_{\mathcal{K}^{s,I^*}}) \models_O A$ by the inductive assumption. By (ii) of Lemma 1, if A is nonmonotonic then we have $I^* \models_O A$ since $\pi'(A)$ is monotonic, and $\gamma_{[\pi'(\mathcal{K})]^{s,I}}^k \models_O \pi'(A)$ implies $I \models_O \pi'(A)$.
- $I^* \not\models_O B$ for any $B \in Neg$ by Lemma 1.

Thus we have that $p \in lfp(\gamma_{\mathcal{K}^{s,I^*}})$.

(\supseteq) We prove this direction by showing that $\gamma_{\mathcal{K}^{s,I^*}}^k \subseteq I$ for any $k \geq 0$.

Base: It is trivial for $k = 0$.

Step: Suppose it holds for the case k . Let us consider the case $k + 1$. For any atom $p \in \gamma_{\mathcal{K}^{s,I^*}}^{k+1}$, there exists a rule $(p \leftarrow Pos, not Neg)$ in P such that

- $\gamma_{\mathcal{K}^{s,I^*}}^k \models_O A$ for any atom and monotonic dl-atom $A \in Pos$, and $I^* \models_O A$ for any nonmonotonic dl-atom in Pos , and
- $I^* \not\models_O B$ for any $B \in Neg$.

It follows that

- In the case A is an atom or monotonic dl-atom, we have $I \models_O A$ by the inductive assumption, by which $I \models_O \pi'(A)$ in terms of Lemma 1. If A is nonmonotonic then $I \models_O \pi'(A)$ by $I^* \models_O A$.
- By Lemma 1, we have $I \not\models_O \pi'(B)$.

Consequently we have $p \in I$. ■

Another interesting observation is that, for the two removed strong answer sets in the examples above, neither is well-supported in the sense of (Shen, 2011), as neither possesses a level mapping justification. One would like to know whether π' removes all answer sets that are not well-supported. The answer is no, as evidenced by the next example. Consider the dl-program \mathcal{K}_1 of Example 4, i.e., $\mathcal{K}_1 = (\emptyset, P_1)$ where P_1 consists of $p(a) \leftarrow not DL[S \ominus p; \neg S](a)$. It is not difficult to see that \mathcal{K}_1 has two strong answer sets, \emptyset and $\{p(a)\}$, and the latter is not well-supported. Now $\pi'(\mathcal{K}_1) = (\emptyset, \pi'(P_1))$ where $\pi'(P_1)$ consists of

$$\begin{aligned} p(a) &\leftarrow not DL[\neg S \oplus \bar{p}; \neg S](a), \\ \bar{p}(a) &\leftarrow not DL[S' \oplus p, S'](a). \end{aligned}$$

It can be verified that both $\{\bar{p}(a)\}$ and $\{p(a)\}$ are strong answer sets of $\pi'(\mathcal{K}_1)$. That is, the strong answer set $\{p(a)\}$ that is not well-supported is retained by π' . Therefore, the translation π' cannot be used as a means to interpret a dl-program under the strongly well-supported semantics.

Continuing the above example by considering the FLP-semantics, we note that \emptyset is the unique FLP-answer set of \mathcal{K}_1 , and the reader can verify that both $\{\bar{p}(a)\}$ and $\{p(a)\}$ are FLP-answer sets of $\pi'(\mathcal{K}_1)$. While $\{\bar{p}(a)\}$ corresponds to the FLP-answer set \emptyset of \mathcal{K}_1 when restricted to HB_{P_1} , the FLP-answer set $\{p(a)\}$ of $\pi'(\mathcal{K}_1)$ has no corresponding FLP-answer set of \mathcal{K}_1 . This shows that extra FLP-answer sets may be introduced by π' .

The next example shows that the translation π' may remove weakly well-supported answer sets. Recall the dl-program $\mathcal{K} = (\emptyset, P)$ where $P = \{p(a) \leftarrow DL[S \odot p, S \ominus p; \neg S](a)\}$. It can be verified that $\{p(a)\}$ is a weakly well-supported answer set of \mathcal{K} (it is also strongly well-supported simply because there is no negative dl-atom in the rule). The π' translation results in

$$\begin{aligned} p(a) &\leftarrow DL[S \odot p, \neg S \oplus \bar{p}; \neg S](a), \\ \bar{p}(a) &\leftarrow not DL[S' \oplus p, S'](a). \end{aligned}$$

It is clear that $\pi'(\mathcal{K})$ has no strong answer sets. Thus, the translation π' is too strong for the weakly well-supported semantics.

To summarize, the translation π' defined for the well-founded semantics of dl-programs is too strong for the strong answer set semantics, and for the FLP semantics and well-supported semantics, it is sometimes too strong and sometimes too weak.

5.3 Other embedding approaches

As to embedding dl-programs into other formalisms that integrate ontology and (nonmonotonic) rules for the semantic web, there are a number of proposals, such as first-order autoepistemic logic (de Bruijn et al., 2008), MKNF knowledge base (Motik & Rosati, 2010), quantified equilibrium logic (Fink & Pearce, 2010), and first-order stable logic programs (Ferraris, Lee, & Lifschitz, 2011; Lee & Palla, 2011). In addition to the differences between default logic and those formalisms,¹³ we also considered the weakly and strongly well-supported answer set semantics of dl-programs, recently proposed by (Shen, 2011).

The discussion below will be based on the strong answer set semantics. As we mentioned at the end of Section 3, the embedding presented by Motik and Rosati works only for canonical dl-programs. By the result of this paper, their embedding can be now extended to normal dl-programs by applying first the translation π . For dl-programs without nonmonotonic dl-atoms, our embedding does not introduce new predicates. The latter is done by the translation of dl-programs into first-order stable logic programs (Ferraris et al., 2011) by Lee and Palla (2011), even for canonical dl-programs.

As commented earlier, the current embedding into quantified equilibrium logic (Fink & Pearce, 2010) works for normal dl-programs only, as the authors adopt a convention that all dl-atoms containing an occurrence of \ominus are nonmonotonic. The embedding of dl-programs into first-order autoepistemic logic in (de Bruijn et al., 2008) is under the weak answer set semantics. For the strong answer set semantics, it is obtained indirectly, by embedding MKNF into first-order autoepistemic logic, together with the embedding of dl-programs into MKNF. Thus it works for canonical dl-programs only.

We also notice that, to relate default theories with dl-programs, Eiter et al. (2008) and Dao-Tran, Eiter, and Krennwallner (2009) presented transformations of a class of default theories, in which only conjunctions of literals are permitted in defaults, to canonical dl-programs (with variables) and to cq-programs respectively. Informally, cq-programs can be viewed as a generalization of canonical dl-programs, where the heads of dl-rules can be disjunctive and queries in dl-atoms can be also (decidable) conjunctive queries over the ontology. Our transformation from normal dl-programs to default theories provides a connection from the other side. Clearly the class of normal logic programs is a subclass of the normal dl-programs. Already Gelfond and Lifschitz (1991) have

13. A discussion of these differences is out of the scope of this paper.

shown that normal logic programs under answer set semantics correspond to default logic. This has now been generalized by our results for normal dl-programs. The work here can be similarly generalized to deal with strong negation as well.

6. Conclusion

In this paper, we have studied how dl-programs under various answer set semantics may be captured in default logic. Starting with the semantics in the seminal paper (Eiter et al., 2008), we showed that dl-programs under weak and strong answer set semantics can be embedded into default logic. This is achieved by two key translations: the first is the translation π that eliminates the constraint operator from nonmonotonic dl-atoms, and the second is a translation τ that transforms a dl-program to a default theory while preserving strong answer sets of normal dl-programs, provided that the given ontology knowledge base is consistent. This proviso is not necessary under translation τ' , which preserves strong answer sets even if the given ontology knowledge base is inconsistent. It also preserves weak answer sets if in addition all dl-atoms occur under default negation. Both translations τ and τ' are polynomial and modular, without resorting to extra symbols.

The translation π depends on the knowledge of whether a dl-atom is monotonic. We have given the precise complexity to determine this property, for ontology knowledge bases in the description logics *SHIF* and *SHOIN*.

The importance of these results is that, for all current approaches to representing strong answer sets, either such an approach directly depends on this knowledge (Fink & Pearce, 2010; Lee & Palla, 2011), or the underlying assumption can be removed, with this knowledge and the translation π above (de Bruijn et al., 2008; Motik & Rosati, 2010).

Furthermore, the translations τ and τ' can be refined to polynomial, faithful, and modular translations τ^* and $\tau^{*'}$, respectively, which capture the recently proposed weakly well-supported semantics for arbitrary dl-programs (Shen, 2011). This is somewhat surprising as the resulting translations are like writing dl-rules by defaults in a native language, enhanced only by normal defaults of the form $\frac{\neg p(\vec{c})}{\neg p(\vec{c})}$. Apparently, the key is that the iterative definition of default extensions provides a free ride to the weak well-supportedness based on a notion of level-mapping, but not to the strong well-supportedness. This is an interesting insight. One would expect bigger challenges in representing the same semantics in other nonmonotonic logics.

For the class of dl-programs that mention no constraint operator, i.e. the class of canonical dl-programs, all major semantics coincide, including strongly well-supported answer sets, weakly well-supported answer sets, FLP-answer sets, and strong answer sets. Thus, the translation τ' can be viewed as a generic representation of dl-programs in default logic. In other words, there is a simple, intuitive way to understand the semantics of (canonical) dl-programs in terms of default logic. Fortunately, many practical dl-programs are canonical as argued in (Eiter et al., 2011). At the same time, we understand the precise complexity of checking monotonicity of a dl-atom, for some major description logics. These results strengthen the prospect of default logic as a foundation for query-based approaches to integrating ontologies and rules. In this sense, default logic can be seen as a promising framework for integrating ontology and rules. We will look into this issue further in future work.

Though we have presented a faithful and modular embedding for dl-programs under strong answer set semantics, the embedding is not polynomial. It remains as an interesting issue whether there exists such a polynomial embedding. In addition, we have shown that τ^* preserves strongly well-

supported answer sets of a highly relevant class of dl-programs, viz. the one in which nonmonotonic dl-atoms do not occur negatively. It remains open whether there exists a faithful, modular embedding for arbitrary dl-programs under the strongly well-supported answer set semantics into default logic.

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Appendix A.

[Proof of Theorem 1 (continued)] (i) To show EXP-hardness for the case of \mathcal{SHIF} knowledge bases, we provide a reduction from deciding unsatisfiability of a given knowledge base O in \mathcal{SHIF} , which is EXP-complete given that deciding satisfiability is EXP-complete (Horrocks & Patel-Schneider, 2003) and EXP is closed under complementation, to checking monotonicity of a dl-atom A relative to a dl-program \mathcal{K} as follows.

Let C be a fresh concept and define the following dl-atom:

$$A = DL[C \ominus p; \top \sqsubseteq \perp]()$$

where p is a fresh unary predicate. Furthermore, let

$$O' = O \cup \{C(o) \mid o \in \mathcal{C}\}$$

where without loss of generality $\mathcal{C} \neq \emptyset$ is the set of individuals occurring in O .

It is clear that if O is unsatisfiable, then A is monotonic relative to $\mathcal{K} = (O', P)$, where $P = \{p \leftarrow A\}$ and p is a fresh propositional atom. Recall that A is nonmonotonic w.r.t. O' iff there exist two interpretations I and I' such that $I \subset I'$, $I \models_{O'} A$, and $I' \not\models_{O'} A$. Every interpretation I such that $p(o) \notin I$ for some $o \in \mathcal{C}$ is a model of A relative to O' , and the interpretation $I \cup \{p(o) \mid o \in \mathcal{C}\}$ is not a model of A relative to O' if O is satisfiable. Hence, A is nonmonotonic relative to \mathcal{K} iff O is satisfiable. It follows that the EXP-complete unsatisfiability test reduces to the DL-monotonicity test, and settles the result for the \mathcal{SHIF} case.

(ii) For the case of \mathcal{SHOIN} knowledge bases, we show hardness for $P^{\text{NEXP}} = \text{co-NP}^{\text{NEXP}}$, building on machinery used in (Eiter et al., 2008) for the complexity analysis of strong and weak answer sets of dl-programs with \mathcal{SHOIN} knowledge bases. In the course of this, an encoding of a torus-tiling problem (that represents NEXP Turing machine computations on a given input) into a DL knowledge base satisfiability problem was used. We briefly recall this problem.

A *domino system* $\mathcal{D} = (D, H, V)$ consists of a finite nonempty set D of *tiles* and two relations $H, V \subseteq D \times D$ expressing horizontal and vertical compatibility constraints between the tiles. For positive integers s and t , and a word $w = w_0 \dots w_{n-1}$ over D of length $n \leq s$, we say that \mathcal{D} *tiles* the torus $U(s, t) = \{0, 1, \dots, s-1\} \times \{0, 1, \dots, t-1\}$ with *initial condition* w iff there exists a mapping $\tau: U(s, t) \rightarrow D$ such that for all $(x, y) \in U(s, t)$: (i) if $\tau(x, y) = d$ and $\tau((x+1) \bmod s, y) = d'$, then $(d, d') \in H$, (ii) if $\tau(x, y) = d$ and $\tau(x, (y+1) \bmod t) = d'$, then $(d, d') \in V$, and (iii) $\tau(i, 0) = w_i$ for all $i \in \{0, \dots, n\}$. Condition (i) is the *horizontal constraint*, condition (ii) is the *vertical constraint*, and condition (iii) is the *initial condition*.

Similar as (Eiter et al., 2008), we use the following lemmas.

Lemma 14 (cf. Lemma 5.18 and Corollary 5.22 in (Tobies, 2001)) *For domino systems $\mathcal{D} = (D, H, V)$ and initial conditions $w = w_0 \dots w_{n-1}$, there exist DL knowledge bases O_n , $O_{\mathcal{D}}$, and O_w , and concepts $C_{i,0}$, $i \in \{0, 1, \dots, n-1\}$, and C_d , $d \in D$, in \mathcal{SHOIN} such that:*

- $O_n \cup O_{\mathcal{D}} \cup O_w$ is satisfiable iff \mathcal{D} tiles $U(2^{n+1}, 2^{n+1})$ with initial condition w ;
- O_n , $O_{\mathcal{D}}$, and O_w can be constructed in polynomial time in n from n , \mathcal{D} , and w , respectively, and $O_w = \{C_{i,0} \sqsubseteq C_{w_i} \mid i \in \{0, 1, \dots, n-1\}\}$;
- in every model of $O_n \cup O_{\mathcal{D}}$, each $C_{i,0}$ contains exactly one object representing $(i, 0) \in U(2^{n+1}, 2^{n+1})$, and each C_d contains all objects associated with d .

Lemma 15 (cf. Theorem 6.1.2 in (Börger, Grädel, & Gurevich, 1997)) *Let M be a nondeterministic Turing machine with time- (and thus space-) bound 2^n , deciding a NEXP-complete language $\mathcal{L}(M)$ over the alphabet $\Sigma = \{0, 1, " "$. Then, there exists a domino system $\mathcal{D} = (D, H, V)$ and a linear-time reduction trans that takes any input $b \in \Sigma^*$ to a word $w \in D^*$ with $|b| = n = |w|$ such that M accepts b iff \mathcal{D} tiles the torus $U(2^{n+1}, 2^{n+1})$ with initial condition w .*

Based on this, (Eiter et al., 2008) showed how computations of a deterministic polynomial time Turing machine with an NEXP oracle can be encoded into evaluating a dl-program, where intuitively dl-atoms correspond to oracle calls. For the problem at hand, we would have to provide an encoding of such a computation into one dl-atom and the check of its monotonicity. To simplify matters, we provide a reduction from the following problem:

NEXP-JC: Given two partial inputs b and b' of the same NEXP Turing machine M such that $|b| = |b'|$, does there exist a joint completion c of the partial inputs of length $|c| = |b| = |b'|$ such that (1) M accepts bc and (2) M does not accept $b'c$.

Lemma 16 *Problem NEXP-JC is complete for $\text{NP}^{\text{NEXP}} (= \text{P}^{\text{NEXP}})$.*

Intuitively, this is seen as follows: the computation path (nondeterministic moves and query answers) of M can be guessed ahead, and after that only a deterministic computation with oracle accesses is made, in which the oracle answers are checked with the guesses. Witnesses for all oracle queries that should answer “yes” can be found in a single NEXP computation, and all queries that should answer “no” can be verified in a single co-NEXP computation (i.e., a NEXP computation for refutation does not accept). The condition $|b| = |b'| = |c|$ can be ensured by simple padding techniques.

Now the reduction of this problem to deciding dl-atom monotonicity is exploiting (and modifying) the torus-tiling problem encoding to DL satisfiability testing quoted above. It has been shown in (Eiter et al., 2008) how to adapt the torus knowledge base such that the initial condition w (encoded by O_w) can be flexibly established by the update string λ of a dl-atom. Intuitively, “switches” were used to “activate” concepts that represent tiles, so that tiles are put in place by the call of the dl-atom.

Using a similar idea, we change O_w . As in (Eiter et al., 2008), assertions

$$C_{i,0}(o_i), \quad i = 0, \dots, n-1$$

are used to introduce individuals o_i for the torus positions $(i, 0)$ that hold the initial condition w encoding a complete input bc resp. $b'c$, where $n = 2m-1$ and $m = |b| = |b'|$; we have $\mathcal{C} = \{o_0, \dots, o_{n-1}\}$. We implement a “switch” that tells whether computation of either (1) bc or (2) of $b'c$ should be considered in a call. For this, we use a concept S and put $S \ominus p$, $S \oplus p$ in the “update” λ of the dl-atom A that we construct, which will effect that given any interpretation I , for each individual o_i either $S(o_i)$ or $\neg S(o_i)$ will be asserted in $O(I; \lambda)$. We pick o_0 (i.e., position $(0, 0)$ of the torus, which is “identified” by the concept $C_{0,0}$) and install on it the switch between case 1) and 2): if $S(o_0)$ is true, we evaluate case 1), else case 2). To “prepare” the part of the initial condition encoding b resp. b' , we use axioms

$$\begin{aligned} C_{0,0} \sqcap S &\sqsubseteq B, \\ C_{0,0} \sqcap \neg S &\sqsubseteq \neg B, \end{aligned}$$

where B is a fresh concept (intuitively, a flag indicating case 1), i.e., b), and an axiom

$$B \sqsubseteq \forall east.B$$

where *east* is a role already defined in $O_n \cup O_{\mathcal{D}}$ which links position (i, j) to $(i+1, j)$, for all i and j ; in combination with the above axioms, it effects that when evaluating a dl-atom w.r.t. an interpretation I , in every model of $O(I; \lambda)$ either all elements e_i at “input” positions are labeled with B or all are labeled with $\neg B$. Depending on the B -label, we then assign e_i the right tile from the initial condition for b (label B) respectively for b' (label $\neg B$):

$$\left. \begin{aligned} C_{i,0} \sqcap B &\sqsubseteq C_{w_i} \\ C_{i,0} \sqcap \neg B &\sqsubseteq C_{w'_i} \end{aligned} \right\} \quad i = 0, \dots, m-1,$$

where w_i (resp. w'_i) is the i -th tile of w (resp. w'). Intuitively, the case of label B is for input I' that is “larger” than input I for label $\neg B$; for the former, we must have $p(o_0) \in I'$ and for the latter $p(o_0) \notin I'$; the value of $p(o_i)$, $i > 0$, does not matter, so we can assume it is the same in I and I' . For I' we do the NEXP test, and for the “smaller” I we do the co-NEXP test. If both succeed, we have a counterexample to monotonicity.

It remains to incorporate the guess c for the completion of the input. This guess can be built in by using concepts S_d such that $S_d(o_i)$ intuitively puts tile d at the position i in the initial condition (where $i = m, \dots, n-1$ runs from the first position after b (resp. b') until the last position of the fully completed input bc (resp. $b'c$), viz. $n-1$). In the input list λ of the dl-atom A , we put

$$S_d \ominus p_d, S_d \oplus p_d \quad d \in D$$

where p_d is a fresh unary predicate (D is the set of tiles). Similar as above, this will assert for each individual then either S_d or $\neg S_d$.

We then add axioms which put on tiles as follows:

$$\left. \begin{aligned} C_{i,0} \sqcap S_d &\sqsubseteq C_d \\ C_{i,0} \sqcap \prod_{d \in D} \neg S_d &\sqsubseteq C_{d_0} \end{aligned} \right\} \quad i = m, \dots, n-1, d \in D$$

where d_0 is some fixed tile; the second axiom puts a default tile if in I no tile has been selected (as if $p_{d_0}(o_i)$ would be in I). If multiple tiles have been selected, then the $O(I; \lambda)$ is unsatisfiable,

and similarly $O(I'; \lambda)$ for each $I' \supset I$. So the interesting case is if exactly one tile has been put on in each “completion” position $i = m, \dots, n-1$ of the initial condition. The selection of tiles is subject to further constraints on tiles at adjacent positions $i-1, i$ from $m, \dots, n-1$ and on the last position, due to the encoding of the machine input into the initial condition in (Börger et al., 1997). Without going into detail here, let $A \subset D^2$ and $F \subset D$ be the sets of admissible adjacent tiles (d, d') and final tiles d , respectively (which are easily determined). We then add axioms

$$C_{i,0} \sqcap C_{d'} \sqsubseteq \forall east^-. \bigsqcup_{(d,d') \in A} C_d, \quad i = m, \dots, n-1, d' \in D,$$

$$C_{n-1,0} \sqsubseteq \bigsqcup_{d \in F} C_d.$$

This completes the construction of O_w . Now let $A = DL[\lambda; \top \sqsubseteq \perp]()$ and $\mathcal{K} = (O, P)$, where $O = O_n \cup O_D \cup O_w$ and $P = \{p(o_0) \leftarrow A\}$. It can be shown that a violation of the monotonicity of A relative to \mathcal{K} is witnessed by two interpretations $I \subset I'$ of form $I' = I \cup \{p(o_0)\}$ such that $I' \not\models_O A$ and $I \models_O A$ and the interpretations encode a joint completion c of the inputs b and b' , meaning that the computation for bc is accepting while the one for $b'c$ is not. As \mathcal{K} and A are constructible in polynomial time from b, b' and M , this proves the result. \blacksquare

Appendix B.

Lemma 17 *Let $\mathcal{K} = (O, P)$ be a dl-program and $I \subseteq HB_P$. Then we have that*

- (i) $\pi_1(I) = \{\pi_p(\vec{c}) \in HB_{\pi(P)}\} \cap lfp(\gamma_{[\pi(\mathcal{K})]^{w, \pi(I)}})$,
- (ii) $\pi_2(I) = \{\pi_A \in HB_{\pi(P)}\} \cap lfp(\gamma_{[\pi(\mathcal{K})]^{w, \pi(I)}})$, and
- (iii) $\gamma_{\mathcal{K}^{w, I}}^k = HB_P \cap \gamma_{[\pi(\mathcal{K})]^{w, \pi(I)}}^k$ for any $k \geq 0$.

Proof: (i) It is evident that, for any atom $\pi_p(\vec{c}) \in HB_{\pi(P)}$, the rule $(\pi_p(\vec{c}) \leftarrow not p(\vec{c}))$ is in $\pi(P)$. We have that

$\pi_p(\vec{c}) \in \pi_1(I)$
 iff $p(\vec{c}) \notin I$
 iff $p(\vec{c}) \notin \pi(I)$
 iff the rule $(\pi_p(\vec{c}) \leftarrow)$ belongs to $w[\pi(P)]_O^{w, \pi(I)}$
 iff $\pi_p(\vec{c}) \in lfp(\gamma_{[\pi(\mathcal{K})]^{w, \pi(I)}})$.

(ii) It is clear that, for any $\pi_A \in \pi_2(I)$, the rule $(\pi_A \leftarrow \pi(not A))$ is in $\pi(P)$ such that $A \in DL_P^?$ and $I \not\models_O A$. Let $A = DL[\lambda; Q](\vec{t})$. We have that

$\pi_A \in \pi_2(I)$
 iff $\pi_A \in HB_{\pi(P)}$ and $I \not\models_O A$
 iff $\pi(I) \not\models_O DL[\pi(\lambda); Q](\vec{t})$ (by (ii) of Lemma 1)
 iff the rule $(\pi_A \leftarrow)$ belongs to $w[\pi(P)]_O^{w, \pi(I)}$
 iff $\pi_A \in lfp(\gamma_{[\pi(\mathcal{K})]^{w, \pi(I)}})$.

(iii) We show this by induction on k .

Base: It is obvious for $k = 0$.

Step: Suppose it holds for $k = n$. Let us consider the case $k = n + 1$. For any atom $\alpha \in HB_P$, $\alpha \in \gamma_{\mathcal{K}^w, I}^{n+1}$ if and only if there is a rule

$$\alpha \leftarrow Pos, Mdl, Ndl, not Neg$$

in P where Pos is a set of atoms, Mdl a set of monotonic dl-atoms and Ndl a set of nonmonotonic dl-atoms such that

- $\gamma_{\mathcal{K}^w, I}^n \models_O A$ for any $A \in Pos$,
- $I \models_O B$ for any $B \in Ndl$,
- $I \models_O B'$ for any $B' \in Mdl$, and
- $I \not\models_O C$ for any $C \in Neg$.

It follows that;

- $\gamma_{\mathcal{K}^w, I}^n \models_O A$ if and only if $\gamma_{[\pi(\mathcal{K})]^w, \pi(I)}^n \models_O A$ by the inductive assumption,
- $I \models_O B$ if and only if $\pi_B \notin \pi(I)$ by the definition of $\pi_2(I)$, i.e., $\pi(I) \not\models_O \pi_B$,
- $I \models_O B'$ if and only if $\pi(I) \models_O B'$, and
- $I \not\models_O C$ if and only if $\pi(I) \models_O \pi(not C)$ for any $C \in Neg$ by Lemma 1.

Thus we have that $\alpha \in \gamma_{\mathcal{K}^w, I}^{n+1}$ if and only if $\alpha \in \gamma_{[\pi(\mathcal{K})]^w, \pi(I)}^{n+1} \cap HB_P$. ■

[Proof of Theorem 4]

(i) We have that

$$\begin{aligned}
 lfp(\gamma_{[\pi(\mathcal{K})]^w, \pi(I)}) &= lfp(\gamma_{[\pi(\mathcal{K})]^w, \pi(I)}) \cap (HB_P \cup \{\pi_p(\vec{c}) \in HB_{\pi(P)}\} \cup \{\pi_A \in HB_{\pi(P)}\}) \\
 &= [HB_P \cap lfp(\gamma_{[\pi(\mathcal{K})]^w, \pi(I)})] \\
 &\quad \cup [\{\pi_p(\vec{c}) \in HB_{\pi(P)}\} \cap lfp(\gamma_{[\pi(\mathcal{K})]^w, \pi(I)})] \\
 &\quad \cup [\{\pi_A \in HB_{\pi(P)}\} \cap lfp(\gamma_{[\pi(\mathcal{K})]^w, \pi(I)})] \\
 &= [HB_P \cap \bigcup_{i \geq 0} \gamma_{[\pi(\mathcal{K})]^w, \pi(I)}^i] \cup \pi_1(I) \cup \pi_2(I) \text{ by (i) and (ii) of Lemma 17} \\
 &= \bigcup_{i \geq 0} [HB_P \cap \gamma_{[\pi(\mathcal{K})]^w, \pi(I)}^i] \cup \pi_1(I) \cup \pi_2(I) \\
 &= \bigcup_{i \geq 0} \gamma_{\mathcal{K}^w, I}^i \cup \pi_1(I) \cup \pi_2(I) \text{ by (iii) of Lemma 2} \\
 &= I \cup \pi_1(I) \cup \pi_2(I) \text{ since } I \text{ is a strong answer set of } \mathcal{K} \\
 &= \pi(I).
 \end{aligned}$$

It follows that $\pi(I)$ is a weak answer set of $\pi(\mathcal{K})$.

(ii) We prove $I^* = \pi(HB_P \cap I^*)$ at first.

$$\begin{aligned}
 I^* &= I^* \cap (HB_P \cup \{\pi_p(\vec{c}) \in HB_{\pi(P)}\} \cup \{\pi_A \in HB_{\pi(P)}\}) \\
 &= (I^* \cap HB_P) \cup (I^* \cap \{\pi_p(\vec{c}) \in HB_{\pi(P)}\}) \cup (I^* \cap \{\pi_A \in HB_{\pi(P)}\}) \\
 &= (I^* \cap HB_P) \cup \pi_1(HB_P \cap I^*) \cup \pi_2(HB_P \cap I^*) \text{ by (i) and (ii) of Lemma 17} \\
 &= \pi(I^* \cap HB_P).
 \end{aligned}$$

Let $I = I^* \cap HB_P$. We have that

$$\begin{aligned}
 lfp(\gamma_{\mathcal{K}^w, I}) &= \bigcup_{i \geq 0} \gamma_{\mathcal{K}^w, I}^i \\
 &= \bigcup_{i \geq 0} (HB_P \cap \gamma_{[\pi(\mathcal{K})]^w, \pi(I)}^i) \text{ by (iii) of Lemma 17} \\
 &= HB_P \cap \bigcup_{i \geq 0} \gamma_{[\pi(\mathcal{K})]^w, \pi(I)}^i \\
 &= HB_P \cap lfp(\gamma_{[\pi(\mathcal{K})]^w, \pi(I)}) \\
 &= HB_P \cap \pi(I) \text{ since } \pi(I) = I^* \text{ is a weak answer set of } \pi(\mathcal{K}) \\
 &= I.
 \end{aligned}$$

It follows that I is a weak answer set of \mathcal{K} . ■